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MATHEMATICAL RESEARCH

IN

DESTROYER-SUBMARINE

ENCOUNTERS

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Prepared by:

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P. O. Box 92
Princeton, N. J.
08540

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13. ABSTRACT

This document is MATHEMATICA's final report under contract N00014-70-C-0307. It describes the results of ~~our~~ mathematical research into destroyer-submarine encounters. The work concentrated principally on a submarine's approach and penetration into a destroyer screen. In our report, a statistical model is developed, the utility of such a model for analyzing sea exercises demonstrated, and the relative success of penetration evaluated for different patrol patterns, speeds, barrier geometries and so forth.

The approach and penetration model uses an instantaneous sonar detection rate or db.-min. model. Its advantages over the so-called cookie-cutter detection model are demonstrated. The results of an investigation comparing the db.-min. model with another detection model are also discussed. Procedures for fitting the instantaneous detection rate model to observed exercise data are provided.

Positioning models were also developed and are described. As a submarine approaches a barrier it is faced with a choice between delaying penetration to possibly gain a more favorable position or initiating penetration to reduce exposure and risk of detection. Where the submarine receives information about destroyer locations, this report models the situation as a stopping rule problem. Where it does not, the situation is modeled with dynamic programming.

Finally, results are obtained about the distribution of a transit point into a barrier under specific assumptions about the destroyer's patrol pattern and the submarine's choice of a transit point. It is shown that the distribution is triangular or approximately triangular in each gap between the destroyers of the screen.

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Security Classification

FOREWORD

Abstract

This document is MATHEMATICA's final report under contract N00014-70-C-0307. It describes the results of our mathematical research into destroyer-submarine encounters. The work concentrated principally on a submarine's approach and penetration into a destroyer screen. In our report, a statistical model is developed, the utility of such a model for analyzing sea exercises demonstrated, and the relative success of penetration evaluated for different patrol patterns, speeds, barrier geometries and so forth.

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Finally, results are obtained about the distribution of a transit point into a barrier under specific assumptions about the destroyer's patrol pattern and the submarine's choice of a transit point. It is shown that the distribution is triangular or approximately triangular in each gap between the destroyers of the screen.

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A. CONSPECTUS OF THE RESEARCH

During the course of this research several aspects of destroyer-submarine encounters have been investigated. The primary work has been concentrated on three vertically integrated models as illustrated in the following figure:

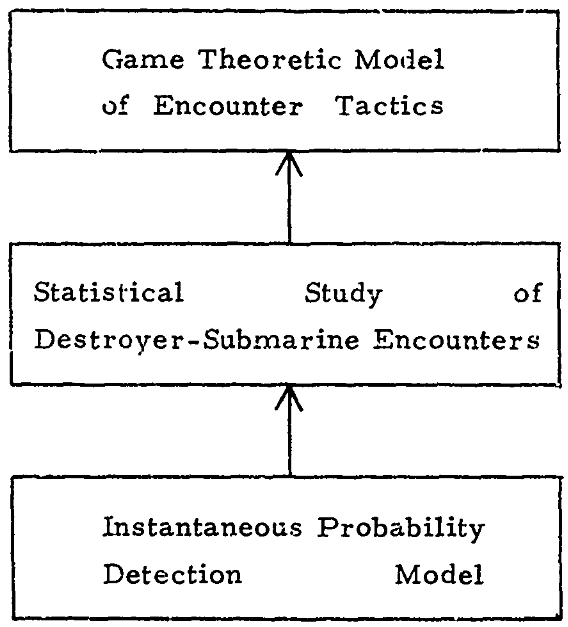

Figure A-1. Encounter Models

At the lowest level is an instantaneous probability detection model. During the time that a submarine is in the vicinity of a destroyer or other sonar receptor, there are continuous transmissions of sound from the submarine. The task of the sonar equipment and of the sonar operators is to separate submarine sound from other noise. The length of exposure at various noise levels determines the probability of detection. Since the noise level is a continuously varying phenomenon, it is necessary to use certain concepts of calculus.

Let us define $N(t)$ as the signal excess over background noise at time t and let us define the instantaneous detection rate $\lambda(t)$ such that in a short interval of time from t to $t + \Delta t$, the probability of detection is approximately $\lambda(t)\Delta t$. The value of $\lambda(t)$ is defined exactly by letting Δt approach zero. The work on this project has shown that $\lambda(t)$ may be approximated by $A \cdot N(t) + B \cdot N^2(t)$. Since $N(t)$ can be estimated from data about equipment and sound propagation, the parameters A and B can also be estimated.

It is thus possible in an encounter in which the relative tracks of a submarine and a destroyer are known to determine the probability of detection. This method is much preferred to a "cookie cutter" model whenever it can be used. In this latter form, the outcome simply hinges on whether the closest point of approach is within the circular area of the "cookie cutter;" thus, no probability is associated with the transit.

Various approaches to instantaneous detection models have been taken but ours appears to be the first to form a relationship between signal excess and instantaneous detection rate based on actual encounters. Recently a similar model*, although without the quadratic term suggested above, has been used in analyzing SHAREM exercises.

The second model in the series represents the situation in which a submarine is attempting to penetrate a convoy screen or barrier patrol of destroyers. The patrol patterns of the destroyers and the approach-penetration pattern of the submarine are simulated under

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* COMDESDEVGRU2, ltrser076, 3 August 1970, "Analysis of Data on Destroyer ASW Screening Mission."

various assumptions. Because of the inherent randomness of the ship tracks, several runs are simulated for any one case and for each run the probability of detection is determined with the first model. The average of these detection probabilities forms the encounter probability of detection under the specified assumptions.

The following assumptions are made in all cases. The destroyers (generally three are simulated) are all actively pinging and the submarine is thus able to hear the destroyer before the destroyer can hear the submarine. In addition, the submarine can estimate the bearing, range, and course of the destroyer during this period.

The following assumptions may be varied from case to case. Passive sonars may be spread uniformly over the barrier; if they are, the ratio of passive to active sonars may be specified.

The destroyer's patrol pattern may be chosen from the following three patterns.

(a) Back and forth patrol. The destroyer patrols between the two end-points of the zonal mid-line (E-W). The destroyer may reverse direction during a leg, as determined by drawing a random number. The frequency of such reversals may be specified.

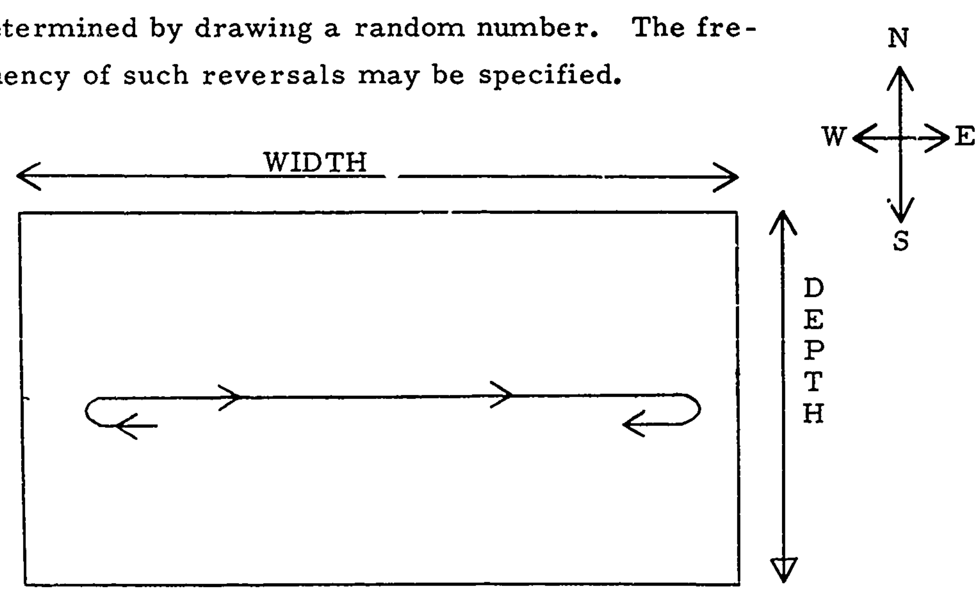
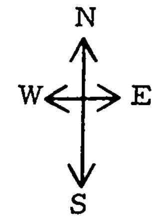


Figure A-2. Back and Forth Patrol

(b) Random segments in random directions within the zone. The destroyer performs straight line segments within the zone as follows:



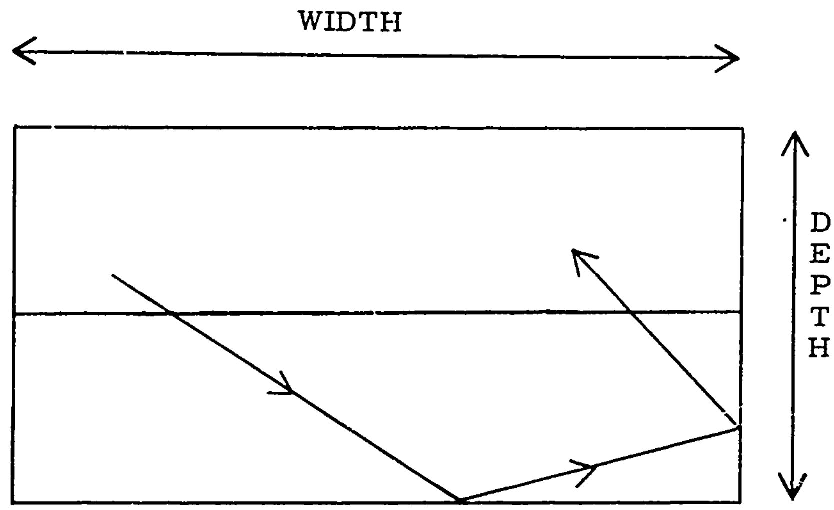


Figure A-3. Random Patrol

At the start of each leg a direction is chosen randomly within $\pm$ 45% of the midline. With some probability the destroyer may reverse course at the start of each leg; otherwise it continues until a boundary is reached.

(c) Zig-zag path at 45° to E-W line. The destroyer moves back and forth along a broken line with each leg at a 45° angle with the midline.

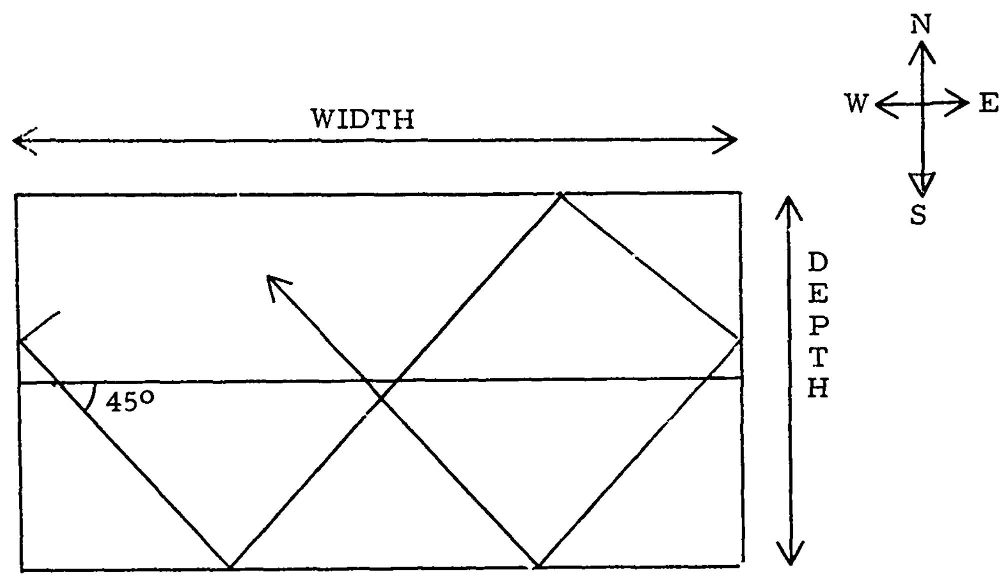


Figure A-4. Zig-zag Patrol

With some probability, the destroyer may reverse course at the start of each leg.

The submarine's approach and penetration pattern is as follows. The submarine approaches from the South and continues its advance until it detects a destroyer. If none is detected, the submarine passes straight through. Otherwise upon hearing the destroyer, it may, depending on an assumed boldness factor, attempt a direct penetration or side-step and attempt to evade the destroyer by penetrating through a gap. In certain situations it may be assumed that the submarine will attempt an end run around the screen.

The use of this model produces tables of detection probabilities under the varied asumptions. The following figure shows a typical array of results for a few of the different assumptions. A complete table might have twenty or thirty entries in each direction.

Submarine Assumptions \ Destroyer Assumptions		Back and Forth	Random	Zig-Zag
Speed 5 kts	Bold	.342	.488	.316
	Normal	.538	.515	.538
Speed 10 kts	Bold	.609	.691	.649
	Normal	.896	.841	.783

Figure A-5. Probabilities of Successful Penetration

These figures are useful in their own right, particularly in reviewing specific encounters, either past or planned. However, the reader may recognize that the assumptions for each side really represent strategies in a game theoretic sense. Thus, the table of results can be regarded as representing a game matrix. In the particular case presented above the submarine's strategy of proceeding at 10 knots and being neither particularly bold nor cautious dominates its other strategies. Consequently the game is easily solved and the destroyer should employ a zig-zag patrol in response to the submarine's actions.

In general, to distinguish the optimal strategies, the game matrix may be solved using linear programming techniques.

Specific issues have arisen in the course of the research that have received special investigation. These additional topics and their relationship to the primary models described above are illustrated in the following figure:

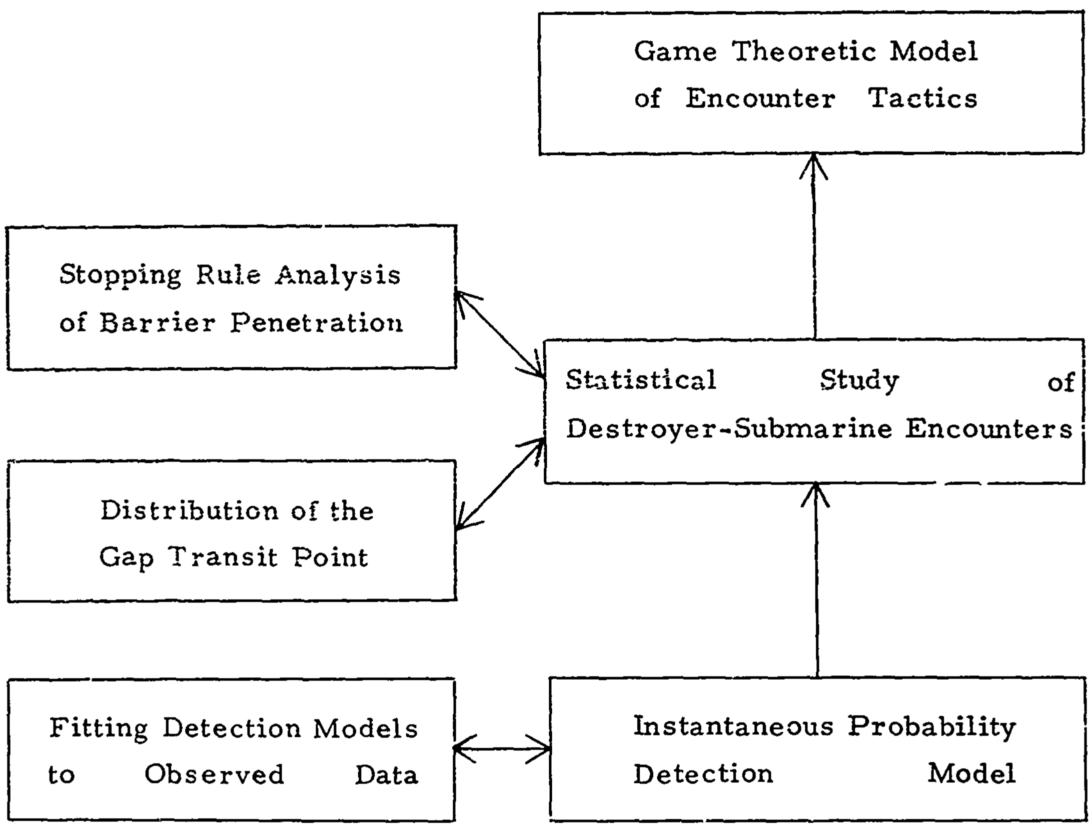


Figure A-6. Encounter and Related Models

The purpose of the stopping rule analysis of barrier penetration is to determine the optimal behavior for a submarine waiting in a "holding pattern" to penetrate a destroyer barrier or screen. At some risk it can wait in an area outside of the detection range but not close enough to obtain information about the destroyers' positions and patrol patterns. The question arises as to the circumstances in which it becomes more advantageous to attempt penetration than to gather more information or wait for a potentially better opportunity. Satisfactory results have been obtained using the theory of stopping rule analysis; all of the information available to the submarine is employed to find its optimal penetration policy.

The probability distribution of the barrier transit point is a collateral result obtained during the study. While not bearing directly on the other models, it is of interest in its own right; a paper on the subject has been prepared and submitted to the Naval Logistics Research Quarterly. In the situation analyzed, destroyers patrol segments of a barrier or screen which may or may not overlap. The submarine is given a chance to pick a gap between the destroyers at an arbitrary point in time. The resulting distribution of the location of the chosen transit point is determined. If two destroyers are present, a triangular distribution results; if three destroyers, an irregular distribution results which nevertheless resembles a triangular distribution in each gap area.

The purpose of developing methods for fitting the instantaneous probability detection model to observed exercise data is to guarantee that the detection model can truly represent the real world. As previously described, the detection rate $\lambda(t)$ is given by $A \cdot N(t)+B \cdot N^2(t)$ where N(t) is the signal excess and A and B are parameters. We have developed a maximum likelihood estimation procedure for determing A and B and an approximate procedure for finding a confidence region on A and B simultaneously. A test for consistency is also developed which makes it possible to test whether the instantaneous detection model adequately explains observed variations in first detection time.

The discussion of destroyer-submarine encounters including a description of the statistical study and its application to game theoretic models is contained in Chapter B. The chapter also reviews several numerical results which have been obtained with the models.

The analysis of barrier penetration problems is contained in Chapter C. Following that, in Chapter D, is a combined discussion of instantaneous probability detection models and of procedures for fitting detection models to observed data obtained from exercises at sea. The last chapter, Chapter E, contains the text of the paper, "The Distribution of the Transit Point in a Submarine vs. Destroyer Game."

B. DESTROYER-SUBMARINE ENCOUNTERS

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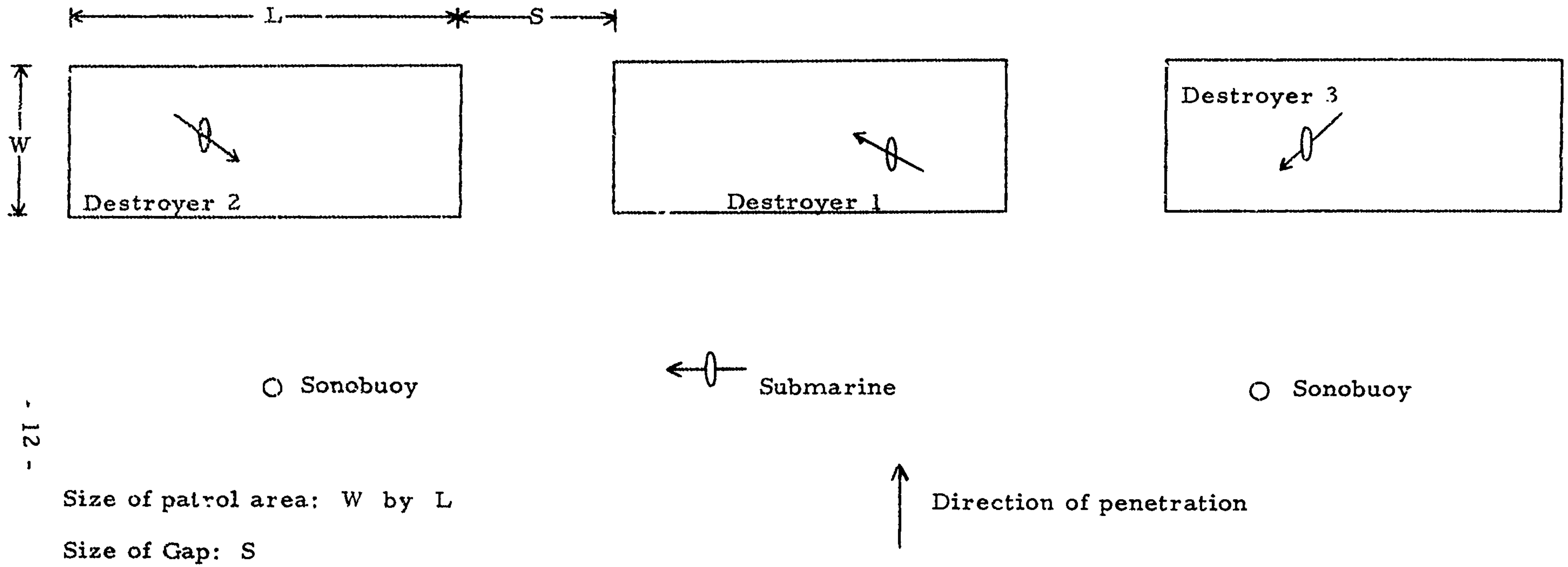
B. DESTROYER-SUBMARINE ENCOUNTERS

1. The Model Scenario

As indicated in the introductory chapter, a statistical model has been developed to assess the probability of undetected penetration by a submarine of a destroyer screen or barrier. This is accomplished by replicating the expected tracks of the ships under specific assumptions about tactics and encounter conditions. Using these tracks and the instantaneous probability detection model, the submarine's probability of survival is determined. Because there are random elements in such encounters, several replications of the encounter are used to obtain an average probability of survival.

The Figure B-1 depicts the patrol areas of the defending destroyers. In each area a destroyer is represented schematically by a vessel-shaped figure, the vector through the vessel representing the direction and speed of movement. The submarine cruising in front of the screen is represented by a similar figure, again with a vector representing direction and speed.

The submarine, approaching from the bottom, may attempt to proceed directly through the screen, may move toward a gap area and proceed directly through, or may cruise or lie dead in the water outside the barrier for a period before attempting a transit. The choice depends on the information available to the submarine's commander and upon his evaluation of that information. The destroyers are assumed always to be in an active sonar mode. Thus the submarine will be able to hear a destroyer at some distance,



Size of patrol area: W by L

Size of Gap: S

Porosity: $\frac{S}{L+S}$

Number of sonobuoys: N

Figure B-1: Geometry of Encounter Scenario

DSUB, which generally will be much greater than the distance, RMIN, at which the destroyer has a nonzero probability of detecting the submarine. The information obtained by the submarine when it is within DSUB of a destroyer allows it to determine the location, course, and speed of the destroyer. We have, for the purposes of this model, assumed that this is the only source of such information to the submarine. A detailed discussion of the way the submarine's response has been modelled is contained in section 3 below on submarine strategies.

The destroyers continually patrol their sectors according to patterns which usually involve random elements. The instantaneous probability detection model described in Chapter D is used for modeling the destroyers' sonar capabilities. In addition to the destroyers, a number of passive sonobuoys may be placed throughout the screen. This tends to discourage submarines from spending long hours in the information gathering region. Various patterns of sonobuoys may be assumed but we have only implemented a uniform type of distribution. As a logical extension, one might consider a distribution which increases the likelihood of exposure for a submarine travelling toward and through gaps. The correct solution of this problem requires the use of game theory.

The detection model used for sonobuoys is the more conventional "cookie-cutter" type in which a submarine is always detected within range SMIN and never outside that range. The value of SMIN is a function of U, the submarine speed, in the usual manner. The distribution of sonobuoys is used with the detection model to obtain a probability of detection for a submarine following a particular path at specified speeds. Because the threat of detection is dispersed, it is less critical whether an instantaneous probability detection model is used and we have found it quite satisfactory in this instance to use a "cookie-cutter" model. To accommodate different scenarios, the number of sonobuoys (expressed as a ratio r to the number of destroyers) is treated as variable. For example, a convoy destroyer screen may have no sonobuoys $(r = 0)$ whereas a stationary destroyer barrier may have many (say, $r = 4$). The details of sonobuoy deployment are described below in Section 4 on sonobuoy deployment.

In the first chapter and above in this section we have indicated in a general way the patrol strategies available to the destroyers. In the following section, we develop this in detail.

2. Description of Destroyer Strategies

The major determinant of a destroyer's strategy is its patrol pattern. The following three patterns have been chosen as an approximation to the methods that are or might be used. There is some redundancy with the text of the first chapter but it is important to have a complete review here.

(a) Back and forth patrol. The destroyer patrols between the two end-points of the zonal mid-line (E-W). The destroyer may reverse direction during a leg, as determined by drawing a random number. The frequency of such reversals may be specified.

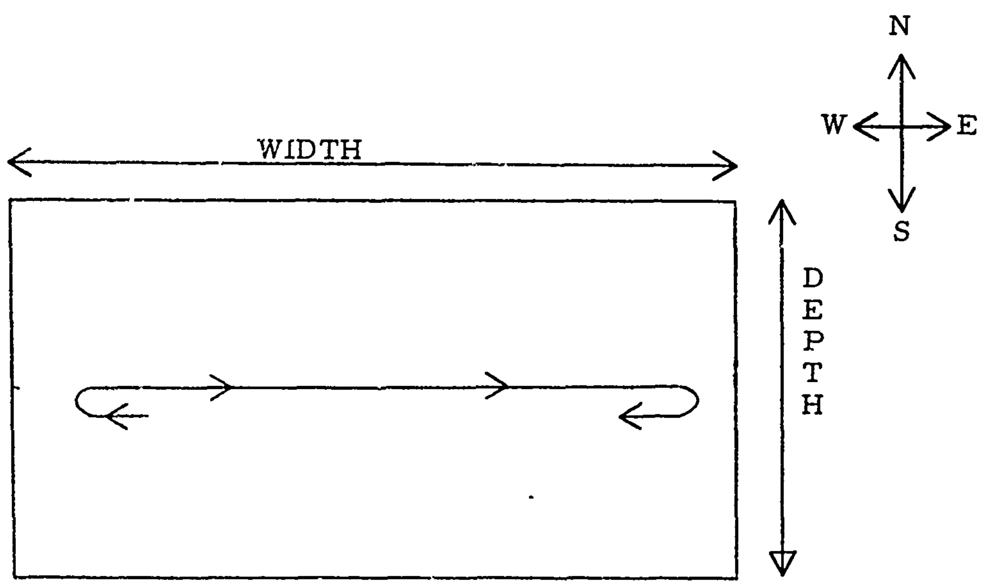


Figure B-2. Back and Forth Patrol

(b) <u>Random segments in random directions within the zone.</u>
The destroyer performs straight line segments within the zone as follows:

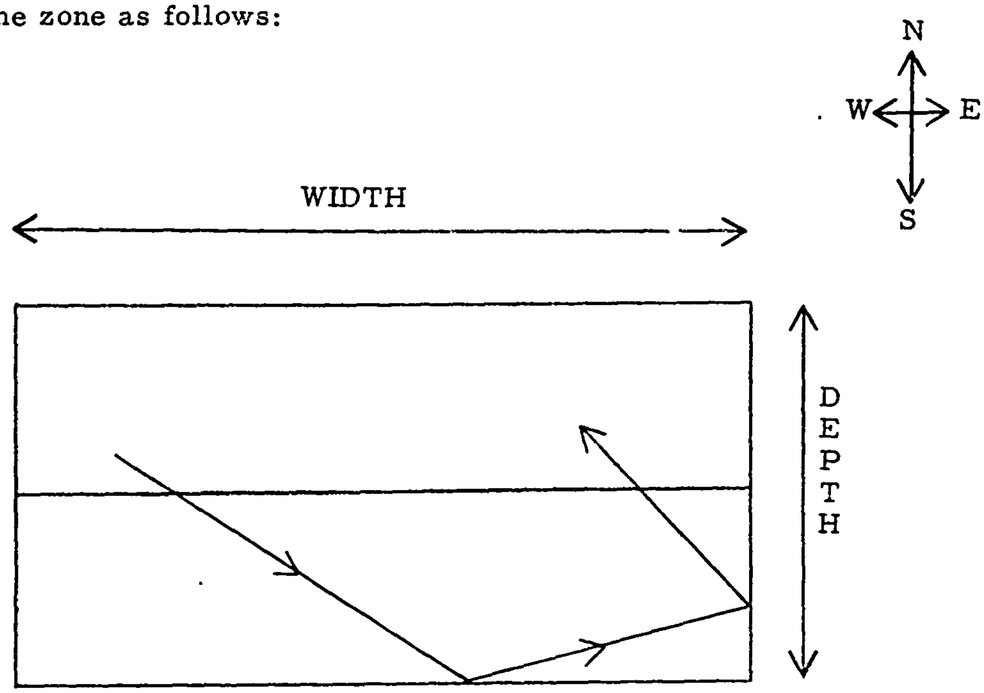


Figure B-3. Random Patrol

At the start of each leg a direction is chosen randomly within $\pm$ 45% of the midline. With some probability the destroyer may reverse course at the start of each leg; otherwise it continues until a boundary is reached. The procedure for selecting each new leg is as follows:

(i) Of the four possible quadrants, the destroyer will usually tend to keep going in the same E-W direction. However, he will occasionally reverse E-W direction. The choice of upper or lower quadrant is made with equal probability, independent of the previous direction.

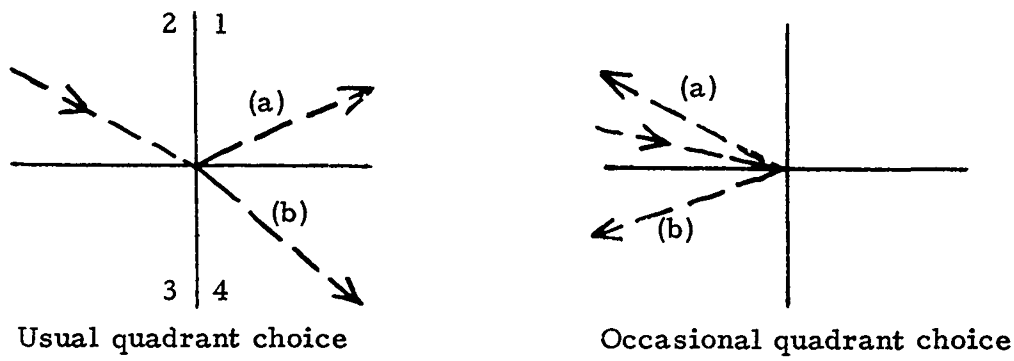


Figure B-4. Choice of Quadrant

(ii) Once the quadrant has been chosen, the angle made with the E-W line is chosen randomly, from the uniform distribution on the range $[0^{\circ}, 45^{\circ}]$.

This follows a suggestion by Fischer [2]. Paths which approach more nearly to the N-S direction "waste time" when seeking a submarine, transitting in the N-S direction. The range of angles chosen gives a reasonable chance of catching a submarine which is lying in wait (or proceeding in an E-W direction).

(iii) Once a path is chosen, it is followed until the boundary of the box is reached. This places obvious restrictions on the next choice of quadrant. A procedure in which path length is chosen randomly (within allowable limits) was tried, and dropped. It used excessive computer time, and contributed little to effective destroyer search.

(c) Zig-zag path at 45° to E-W line. The destroyer moves back and forth along a broken line with each leg at a 45° angle with the midline.

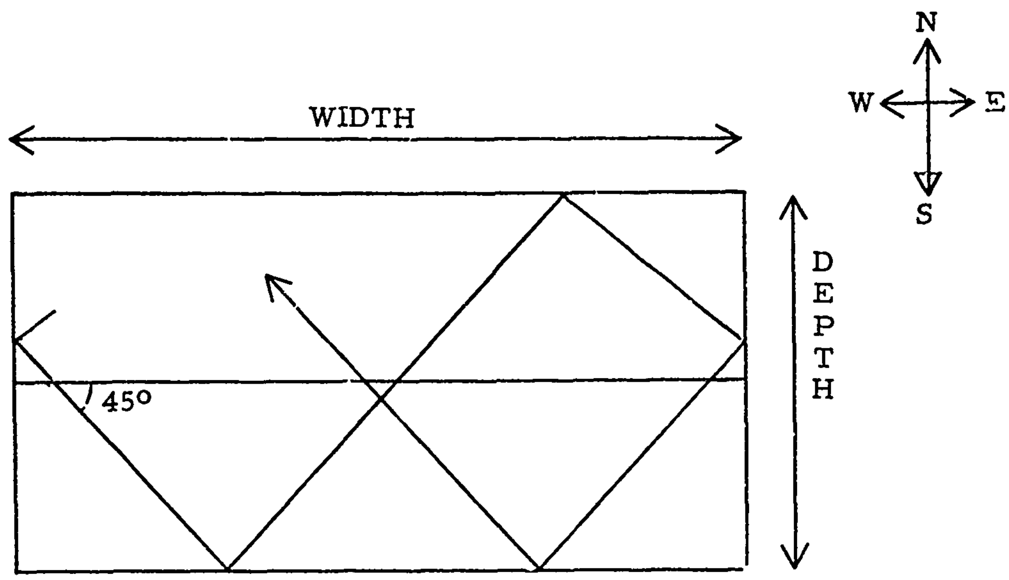


Figure B-5. Zig-zag Patrol

With some probability, the destroyer may reverse course at the start of each leg.

The parameter used to determine the frequency of random reversals (or random changes from quadrants 1, 4 to 2, 3 and vice-versa in case (2) will be called W ("WEIGHT" in the program listing).

W = 1 corresponds to no random reversals. Patterns (1) and (3) are thus completely predictable. Pattern (2) will not be predictable, because the angle with the E-W line is still chosen randomly, but the destroyer will proceed regularly from one end of the barrier to the other. W = .5 corresponds to 50% reversals. W less than .5 is not used, since more frequent reversals than 50-50 would serve no purpose (a number of runs with W less than .5 were made to verify this).

3. Description of Submarine Strategies

The submarine, of course, proceeds due North across the barrier, if it hears no pings. If it hears only one destroyer, it may keep going North, or take a sidestep (E or W) until his Northerly course can be resumed without danger.

The basic choices for the submarine thus are a) to attempt transit through the barrier segment that presents itself as the submarine approaches the barrier, using sidesteps if necessary and maintaining a constant speed; or b) to seek the end of a destroyer segment ("end-run") by determining the larger gap between the three destroyers facing the submarine, proceeding East or West to that gap, and holding to the mid-line of the gap in proceeding North. It may wait dead in the water before proceeding.

The optimal choice will depend on the lambda constant for the destroyer's sonar equipment. For low lambda (poor equipment), a direct transit may be preferable.

If the submarine's speed is relatively slow - in the range 3-10 knots - it may be advisable to ignore a destroyer even if it is heard, provided that the destroyer stays more than the maximum sonar range away. A fast submarine, on the other hand, can afford to take evasive action whenever the predicted closest point of approach (CPA) is less than sonar range.

If the submarine has heard, or hears at present, two or more destroyers, it may attempt a "gap-splitting" strategy. If only two destroyers are heard, it would initially go for the "unprotected" side of the nearer one. ["End-run"] If three are heard, it would split the larger gap. We assume that the submarine keeps records of the Westmost and Eastmost positions of each destroyer heard (and that it can discriminate between them). The end-run case would be converted to gap-split if another destroyer were encountered.

Alternatively, when the submarine hears a second destroyer, it could simply concentrate attention on the closest, or potentially most dangerous one (taking account of the destroyers' current course), and ignore the others. ("No Split" strategy.)

The choice between these two alternative types of strategy will depend on, among other parameters, the spacing between barrier segments chosen by the destroyers. Their object should be to present the submarine with equally desirable alternatives, so that the probability of detection remains constant whichever strategy is chosen.

We have now specified strategy when two ore more destroyers have been heard. Let us return to the case where only one is heard, or when the submarine decides to consider only the most dangerous one (the latter is called the "no split" case).

Upon detecting a destroyer, the following closest points of approach (CPA's) are calculated. It is assumed that the submarine immediately determines the destroyer's course.

(a) CPA_{go} - the resulting CPA if the submarine maintains his present course

(b) CPA_{stop} - the CPA that results if the submarine adopts an Easterly (or Westerly - whichever is better) course.

Both of these CPA's are continuously monitored.

If the submarine is advancing in a Northerly direction, it uses the rules:

If $CPA_{go} \geq$ RMINP, then continue the Northerly course

If $CPA_{go} <$ RMINP, but $CPA_{go} \geq$ GNGFAC $\cdot$ CPA_{stop}, also continue Northerly. The value of GNGFAC (Go/No-Go Factor) is a submarine decision factor which is usually set to about three quarters.

Otherwise, adopt the better Easterly or Westerly direction.

If the submarine is in the East-West mode, it uses the rule:

As soon as $CPA_{go} \geq$ RMINP, begin a Northerly course.

We define a parameter RMINP as follows: the submarine maneuvers taking sidesteps when necessary so as to keep, if possible, the predicted CPA with the destroyer greater than the distance RMINP. Unless his speed is very fast, he will not always be successful.

We have chosen to relate RMINP to RMIN, the destroyer's sonar range or maximum distance at which it has a positive probability of detecting the submarine. This seems reasonable because as sonar is improved the usual consequence is to enlarge the patrol sectors and, in general, to scale up all of the distance parameters in the model. Thus we define

$$RMINP = CC \cdot RMIN$$

where CC is a constant specified initially and, possibly, modified during the encounter. Values of CC greater than one are progressively more cautious while values less than one are progressively more bold. A value of CC equal to zero means the destroyers will be ignored.

Suppose the submarine comes closer to the destroyer, than RMINP. This is an indication that the value of RMINP chosen originally was too cautious. In this event, then CC is reduced by a factor α less than one; thus the submarine becomes increasingly bold. If the destroyer again approaches within the new RMINP, it is reduced by α again, and so on, until the sumbarine becomes sufficiently bold to attempt transit on the next favorable opportunity. This factor α we call the learning factor. In the principal runs obtained during this study α was set to one-half. Table VII represents a brief study of the effects due to variation of α. Further research might profitably analyze the sensitivity of this number and attempt to relate it to observed behavior. It would be especially interesting to find wheather α and CC are correlated.

The introduction of these two factors is an attempt to assist the development of a model of a submarine's behavior in this type of encounter scenario. The factors reflect a commander's initial and subsequent assessment of all threats which exist if a transit is not attempted immediately or if he stops during penetration to take evasive action.

4. Description of Sonobuoy Strategies

Passive sensors are assumed to have the following characteristics:

(i) They are distributed randomly over a specified area; for simplicity, the area used is the full barrier with the addition of strips along the North and South edges of width RMIN.

(ii) The mode of detection is cookie-cutter, with radius $\overline{R}$; that is, detection is certain if the submarine ever comes within range $\overline{R}$ of any passive detector.

(iii) The value of $\overline{R}$ depends on the speed of the submarine. The formula used is given in Appendix B-III.

The number of passive sensors is expressed as ratio to the number of active destroyers. This is appropriate since the sonobuoys are associated with the destroyer sectors and constant ratios imply constant sector strength. The ratio is denoted by r. If it is assumed that each destroyer acts simultaneously (and continuously) as a passive detector, then r must be at least 1. However, values of r = 0, and .5, are included in our analysis, for completeness, as well as r = 1., 2., 4., 6.

The probability of surviving detection by passive sensors is computed, using a simple formula given in Appendix B-III. This survival probability is multiplied by the previously obtained survival probability relating to active sonars.

5. Results of the Model

The value of the present model using the db. -min. concept of instantaneous probability of detection is shown in the program's sensitivity to a large number of control parameters. The simulation program was also run with a "cookie-cutter" detection model to contrast the significantly greater sophistication available with the db-min detection model. Optimal strategic choices change as the condition of the destroyers equipment or the background noise varies. Such differences can not be simulated in any acceptable manner using cookie-cutter detection concepts.

Since the model incorporates a large number of parameters, complete testing under all combinations of parametric variation was computationally unfeasible. Instead, situational parameters, (e. g., barrier geometery) were set equal to standard values and the strategic parameters (e. g., submarine speed and destroyer patrol pattern) were varied. The effect of variation in the situational parameters was investigated by allowing various subsets of the parameters to vary singly for a limited number of strategy choices. Careful examination of the results of the simulation runs gives great insight into the complex interaction between strategic and situational parameters.

The results in the first five sets of tables were obtained holding the following situational parameters fixed at the values indicated:

RMIN	=	10.	=	Destroyer maximum sonar range
DSUB	=	20.	=	Submarine maximum sonar range
BARWID	=	20.	=	Half width of destroyer patrol box
BARDEP	=	5.	=	Half depth of destroyer patrol box
HORFCT	=	2.5	=	Horizontal spacing factor between destroyers
VERFCT	=	0.	=	Vertical spacing factor between destroyers
DESTNO	=	3	=	Number of destroyers
RDFAC	=	0.5	=	Reduction factor for RMINP
GNGFAC	=	0.75	=	Go/No-go decision factor
V	=	17.	=	Destroyer speed (kts.)

Game matrix entries were derived only as necessary to locate the solution strategies. The following four basic and illustrative game matrices will be discussed in detail:

Table I-a: Probability of undetected barrier transit under "average" equipment or sea state conditions with no passive sensors deployed ($\lambda = 0.1$ db - min, $r = 0$).

Table II-a: Probability of undetected barrier transit under "good" equipment or sea state conditions with no passive sensors deployed ($\lambda = 1.0$ db - min, $r = 0$).

Table III-a: Probability of undetected barrier transit under "average" equipment or sea state conditions with four passive sensors deployed per active sonar ($\lambda = 0.1$ db - min, $r = 4$).

Table IV-a: Probability of undetected barrier transit under "good" equipment or sea state conditions with four passive sensors deployed per active sonar ($\lambda = 1.0$ db - min, $r = 4$).

Tables I-b, II-b, III-b, IV-b tabulate the game theoretic solutions for the corresponding game matrices as detailed above. The derivation of strategy solutions requires the following reasonable premises:

i) The destroyer knows the maximum submarine speed

ii) The destroyer chooses a patrol pattern and a course reversal frequency

iii) The submarine knows the choice of patrol pattern and course reversal rate made by the destroyer

iv) The submarine chooses a speed (less than or equal to his maximum speed) and a caution factor.

These choices are denoted by:

IZIG = destroyer patrol pattern

WEIGHT = destroyer course reversal frequency
(WEIGHT = 0.5 implies 50% course reversal
0.7 implies 30% course reversal
1.0 implies no course reversal)

U = submarine speed

CC = submarine caution factor.

The results of the simulation runs when the program was set to use a cookie-cutter detection model are exhibited in Table V-a. Table V-b lists the corresponding optimal strategies based upon this model of detection. Comparisons between Tables I through IV and Table V demonstrate clearly and effectively the importance of the db - min or instantaneous probability of detection concept.

The results displayed in Tables VI through X were derived from the model by holding the strategic parameters fixed at one of two sets of values. Tests were made to demonstrate the sensitivity of the simulation program to a number of critical and/or interesting parameters.

Table VI presents the effect of variation in DSUB, the submarine's maximum sonar range. The next two tables (VII and VIII) tabulate the results of an examinatiation into two of the factors which govern the submarine's movement through the patrol barrier. Table VII displays the results of changes in RDFAC, the factor by which the desired avoidance distance (RMINP) is reduced whenever the destroyer's motion violates this bound. In Table VIII are listed the simulation results due to variation of the parameter GNGFAC; this parameter controls the submarine's decision whetner to go North or to wait (see Appendix B-I). The geometric aspects of the barrier patrol box are the subject of Tables IX and X. Table IX displays the effect that BARDEP, the half depth of the patrol box, has on the probability of undetected transit. The results of variation in the gap size are summarized in Table X.

Probability entries in all the tables of this section denote the probability that the submarine will achieve undetected transit of the patrol barrier starting from a distance RMIN in front of the destroyer patrol box and ending at a distance of RMIN past the patrol box. It must be emphasized that the results are subject to rather large standard deviations. For some probabilitic entries the confidence interval at the 90% level were derived. Where calculated, these are displayed in the form $\pm.0x$. The conclusions drawn from the data are therefore tentative and should be subjected to further investigation. However, the preliminary inferences of this section are believed to be indicative of the correct evaluation of the ASW problem.

Submarine strategies are denoted by

U_{MAX} = maximum submarine speed in knots

U_{OPT} = optimal submarine speed in knots

CC = submarine caution strategy, where <u>increasing</u> values represent <u>increasing</u> caution.

Destroyer strategies are denoted by

IZIG = destroyer patrol pattern

WEIGHT = destroyer frequency of course reversal, where <u>increased</u> course reversal is reflected in <u>lower</u> values of this parameter.

All tables will be found at the end of this section beginning on page 44.

Tables I-a and I-b

The entries in Table I-a were derived assuming "average" equipment and sea states with no passive sensors deployed. The solutions to this game matrix are presented in Table I-b. The table serves as a good illustration of the sensitive nature of the ASW problem. As the maximum submarine speed varies no overall structure is developed in the optimal strategy choices. Only when the submarine is capable of speeds over 10 knots does any pattern develop: the destroyer should choose the method of random patrol and reverse course with frequency 0.5. This strategy may be interpreted to mean that for relatively high submarine speeds, the destroyer patrol should be as unpredicable as possible. The submarine's probability of achieving undetected penetration through the barrier are so high that the destroyer's best strategy is to use chance to trap the submarine within sonar range.

Tables II-a and II-b

In these tables "good" equipment and/or sea states are assumed. Again no passive sonars are deployed. Of particular interest are the solution strategies shown in Table II-b. The choices display significant differences when compared to Table I-b. These can be attributed to the improved value of λ. For low values of U_{MAX} the probability of no detection under optimal strategy choices is considerably reduced:

U_{MAX}	Pr (no detection) λ = 1.0 db - hrs.	Pr (no detection) λ = 0.1 db - hrs.	Differences
3	.010	.295	.285
5	.118	.511	.393
7	.254	.621	.367
10	.514	.809	.295

When the destroyers face a submarine capable of higher speeds the effective differential of better equipment and/or sea states falls off rapidly:

U_{MAX}	Pr (no detection) λ = 1.0 db - hrs.	Pr (no detection) λ = 0.1 db - hrs.	Differences
13	.677	.865	.188
15	.813	.921	.108
17	.843	.921	.078

The game solutions show little regularity except for CC, the caution factor, which is generally equal to 1. This implies the submarine should neither be very bold nor very cautious. The optimal frequency of course reversals (WEIGHT) tends to indicate that trapping the submarine is a preferred strategic choice at higher submarine speeds. By trapping we mean that the submarine, having begun a transit attempt as the destroyers moved away, is caught when one of them unexpectedly reverses course. The best speed for the submarine is in every case equal to the maximum allowable. The probable interpretation of this result is that the submarine does best to transit the barrier as quickly as possible; the submarine thereby reduces the possible time it may be exposed to the destroyers' sonar to a minimum, not worrying about the extra noise generated at high speed.

Tables III-a and III-b

The data in these tables are the result of running the simulation program under the hypothesis of "average" equipment and/or sea state. However passive sonar detectors were simulated in addition. A ratio of four passive sensors per active sensor was used.

The addition of passive detectors had its most significant effect at average and high submarine speeds (assuming optimal strategies are chosen):

U_{MAX}	Pr (nc detection) with passive detectors	Pr (no detection) with no passive detectors	Difference
17	.304	.921	.617
15	.304	.921	.617
13	.304	.865	.561
10	.304	.809	503
7	.303	.621	.318

When the destroyers face submarines with low maximum speeds the range for possible improved detection is limited and deployment of passive detectors is not as efficient as improved equipment.

U_{MAX}	Pr (no detection) with passive detectors	Pr (no detection) with no passive detectors	Difference
5	.288	.511	.223
3	.178	.295	.117

The optimal strategic choice of the destroyer stabilizes for submarine speeds of 7 knots or greater. The submarine's optimal strategic choice becomes fixed if speeds of 10 knots or greater are possible. The rate of destroyer course reversals is less than in previous tables where no passive detectors were deployed. The implication is that passive sonars provide the defenders with sufficient extra listening capability, reducing the need for trapping the submarine. More emphasis can be placed by the destroyers on a systematic search.

The most notable feature of Table III-b is the limitation on the submarine's optimal speed to 10 knots. The significant differences between optimal and maximum available speeds is explained as follows: the fact that passive detectors are deployed randomly, means the submarine can not predict their location. (The destroyers generate enough noise under our assumption that any destroyers course and position can be determined when submarine and destroyer's course and within a distance of DSUB, which equals 20 n.m. in these tables.) Thus in order that the submarine may retain some chance of undetected transit, his speed must be restrained to the threshold value of 10 knots (though it is interesting to note that a threshold value of 5 knots yields a probability of no detection which does not differ within any statistical significance.)

Tables IV-a and IV-b

In these tables the probabilities of undetected transit and the associated game theoretic solutions are considered under conditions of " good" equipment and sea state with the deployment of passive sonars in a ratio of 4:1 to active sensors. The reductions in the probability of undetected transit at the game solutions is dramatic:

U	Pr (no detection) $\lambda = 1.0$ db - hrs. with passive detectors	Pr (no detection) $\lambda = 0.1$ db - hrs. no passive detector	Difference
3	.010	.295	.285
5	.069	.511	.442
7	.117	.621	.504
10	.173	.809	.636
13	.173	.865	.692
15	.173	.921	.748
17	.182	.921	.739

The submarine's optimal speed is essentially limited to 7 knots. The case where the submarine's maximum allowable speed is 17 knots (which equals the destroyer speed) may be considered in the following light. At this threshold value the submarine's best policy is to choose his time and transit path carefully, thereby completely avoiding active sonars while taking the chance that few passive sonars are within sonar range.

This conclusion is supported by the fact that a caution factor of 2 is optimal and that the mean path length of the submarine transit is significantly greater than usual (50 n.m. versus 38 n.m.), i.e. the submarine has spent considerable time moving East to West before proceeding North. The best destroyer policy is to adopt a pattern with no course reversals, the randomization of motion angles in pattern 2 being sufficiently unpredictable in these situations.

Tables V-a and V-b

The results derived from running the simulation program with the relatively unsophisticated cookie-cutter model of detection are displayed in these tables. The inadequacy of using this method to simulate the destroyers' sonar gear is obvious in that there is no satisfactory match between optimal strategies under the two modes. The probabilities of undetected penetration of the barrier most nearly match the db-min model with $\lambda = 1.0$ db-hrs. However when the submarine is restricted to low maximum speeds the differences are highly significant and too large to be acceptable.

Solutions based upon the cookie-cutter model can therefore be seriously misleading. In so far as possible the use of this simplistic model of sonar equipment should be avoided in the analysis of future exercises or simulations.

Table VI

The effect of a variation in DSUB, the submarine's maximum sonar range was investigated for two typical strategy choices. The results as displayed in this table indicate that in some cases it may be of advantage to the submarine to artifically limit its detection range to a value less than the maximal range. A possible interpretation of this non-intuitive conclusion is as follows.

In the cases considered destroyer course changes and reversals are a prominent random factor in the pattern of motion the destroyer traces out in time. The submarine's ability to predict the destroyer's path is thereby severely restricted. An optimal value of DSUB less than the maximum possible suggest that the submarine should not base its own movement though the barrier upon predictions of destroyer motion far into the future. Instead, the submarine should limit his predictions to close encounter situations. In these cases the relevant time horizon required is short enough to preclude the greater part of random destroyer course changes. The submarine improves the overall probability of no detection by skirting the dangerous area of the destroyer's sonar range. As the destroyer's ability to detect the submarine improves the optimal value of DSUB can be expected to increase (closer approach becomes more dangerous to the submarine). This deduction is borne out by the fact that when $\lambda = 1.0$ db - hrs the optimal DSUB derived from Table VI is greater.

Table VII

The learning factor α is described in Section 3. This table investigates the sensitivity of the model to changes in the value chosen for α. As long as α is constrained to the values $.5 \leq \alpha \leq .9$ for the case U = 10. n. m. and to the values $.5 \leq \alpha \leq 1.0$ for the case U = 5. n. m. the resultant probabilities of undetected transit display statistically non-significant variation. In general values of $\alpha = .9$ and $\alpha = 1.0$ appear optimal.

Table VIII

This table lists the effects of variation in GNGFAC, the factor which governs the submarines decision whether to go North or to wait (see Section 3). As indicated in the table, at low speeds GNGFAC = 1.0 appears to be clearly optimal. At higher submarine speeds GNGFAC should be reduced to lie between .75 and 1.0, in order that optimality be retained. These values may be interpreted to mean that at low speeds the submarine should choose his opportunity to head through the barrier with greater caution. When U = 10 the submarine can display somewhat less caution, probably due to the fact that his time of exposure to the defender's sonar gear is significantly reduced.

Tables IX and X

The geometric configuration of the destroyer patrol boxes is considered in these tables. Since the depth of the barrier patrol box is of no consequence to a destroyer using the straight back and forth type of patrol (IZIG = 3), only one illustrative case is considered in Table IX where the results of BARDEP variations are listed. The table reveals that for "average" equipment and/or sea states the destroyer should patrol a fairly shallow box, i.e. BARDEP = 2.5 n.m. is optimal. If "good" equipment and/or sea states are obtained, the barrier depth should be increased to an optimal value BARDEP = 7.5 n.m. The probable conclusion to be drawn is that when the destroyer achieves an advantage over the submarine due to improved sonar ability, a larger area can be examined. When conditions are not so favorable the destroyer should limit the search areas North-South component. For the case under consideration these conclusions support and verify the results of Fisher[2].

Table X was prepared according to the following scheme. It is presumed that the center points of the destroyer patrol boxes are to remain 50 n.m. apart. Each value of BARWID will then result in a gap between the patrol boxes. Though the gap may be swept by the destroyers' sonars, they do not enter the gap. If the submarine can determine the location and width of the gap, it is generally to his advantage to transit the barrier by traveling through the gap.

The results indicate that a gap between patrol boxes of 20 n.m. is optimal if no passive sensors are deployed. The optimal gap size is 0 n.m. (i.e. no gap at all) if passive detectors are deployed uniformly over the barrier in a ratio of 4:1. If appears that since the passive detectors (in the present configuration of the model) are randomly distributed throughout the patrol boxes as well as in the gaps, the passive sensors intensify the destroyer's coverage of the boxes. When no passive detectors are present the destroyer does best by concentrating his search effort to a more limited area, with frequent sweeps of his sonar into the gap. The natural difficulties the submarine faces in attempting to determine the gap location accurately serve to limit the submarines ability to effectively utilize the large open space between two destroyers.

Care must be taken in interpreting these tables. Parameters were varied individually, holding all other external variables fixed. The results of varying two parameters simultaneously can not always be predicted from the single variation results. An example is illustrative.

Consider the case U = 10.0, CC = 1, IZIG = 2, WEIGHT = 0.7, λ = 0.1 db-hrs., and r = 0. Table IX indicates that a barrier depth of BARDEP = 2.5 is optimal. Table X allows the conclusion that a gap size of 20 n.m. is a best choice for the destroyer. However when the gap is 20 n.m., BARDEP = 2.5 is no longer optimal as illustrated in the following table.

BARDEP	Pr (no detection)
7.5	.809
5	.693
2.5	.805

With a gap of 20 n.m., BARDEP = 5. is optimal! Extensive research is necessary in order to determine the optimal values for all parameters that are incorporated into the model's structure simultaneously.

Several other results were obtained from the model. These findings are of a more general nature and are graphed below. In figure B-6, the probability of a submarine achieving undetected penetration of the patrol barrier is graphed as a function of the maximum submarine speed for several values of r (the passive:active sensor ratio). This data is derived from the case U = 10., CC = 1, IZIG = 2, WEIGHT = 0.7, λ = 0.1 db - hrs.

Figure B-7, shows the optimal submarine speed as a function of r, the ratio of passive to active detectors. The variable parameters were set as follows: CC = 1, IZIG = 2, WEIGHT = 0.7, λ = 1.0 db - hrs.

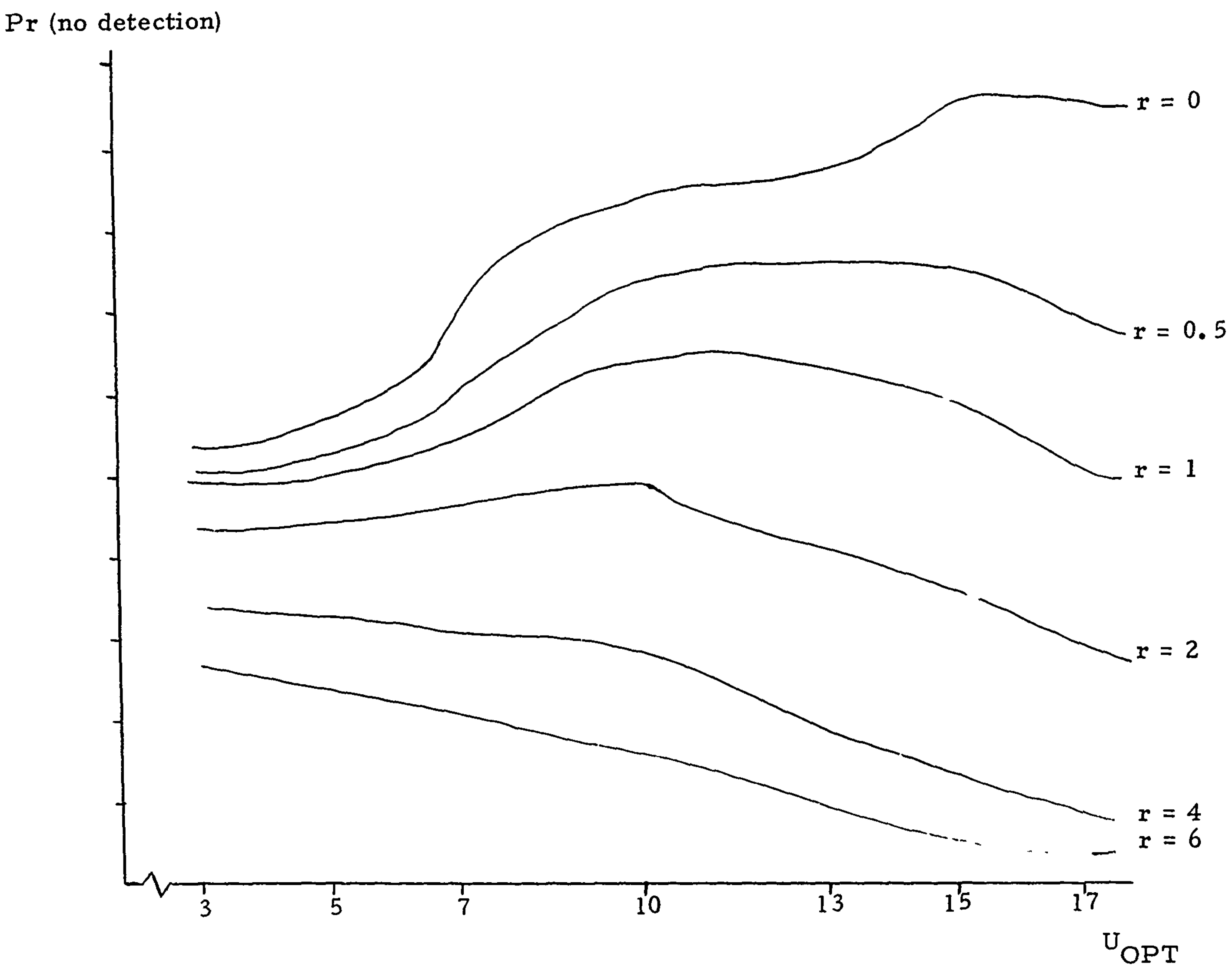


Figure B-6. Probability of no detection as a function of speed for various ratios of passive to active sonars.

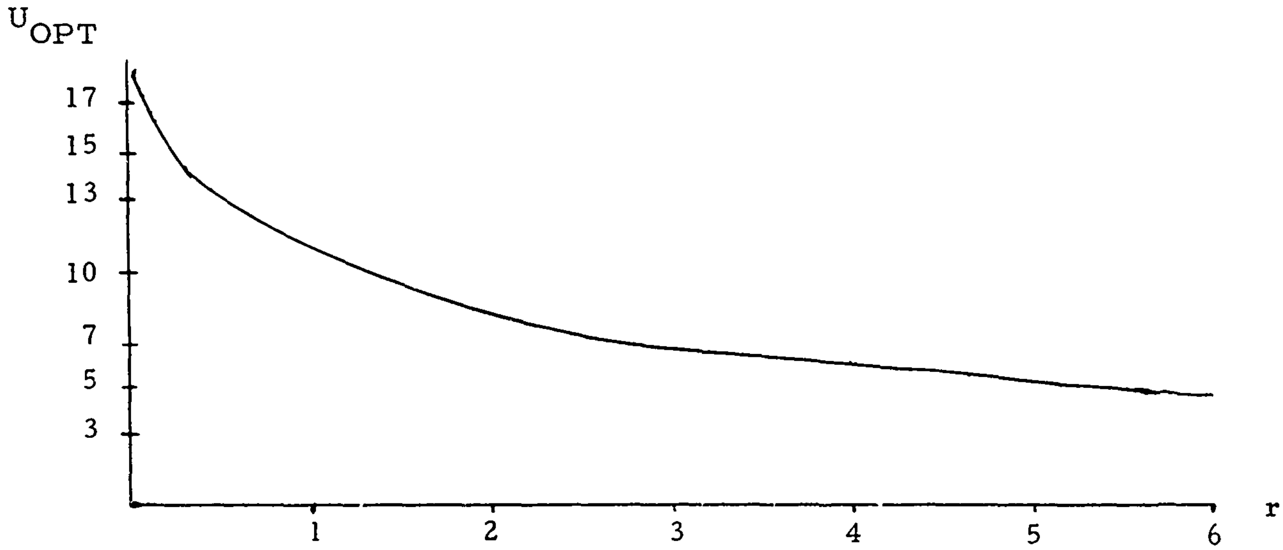


Figure B-7. Optimal Submarine Speed

Table I-a

No passive sonars

Detection rate: 0.1 db - hrs.

Sub U	Destr IZIG CC / W	1 - ZIGZAG 5	1 - ZIGZAG .7	1 - ZIGZAG 1.	2 - 'RANDOM' .5	2 - 'RANDOM' .7	2 - 'RANDOM' 1.	3 - STRAIGHT LINE .5	3 - STRAIGHT LINE .7	3 - STRAIGHT LINE 1.
3	0		.321±.05	.233±.03						
	1	.510±.07	.308±.05	.234±.03	.506±.06	.538±.06	.523±.05	.452±.06	.370±.06	.439±.06
	2		.289±.05	.295±.03						
5	0			.316±.04	.452±.06	.488±.06		.335±.04	.342±.05	
	1	.661±.06	.654±.05	.538±.05	.546±.05	.515±.05	.674±.05	.511±.05	.538±.05	.586±.07
	2			.398±.04	.408±.04	.680±.04		.492±.05	.540±.04	
7	0			.621±.05	.517±.06	.552±.07			.537±.04	
	1	.801±.05	.695±.06	.610±.06	.410±.03	.654±.05	.672±.05	.709±.06	.617±.05	.797±.05
	2			.533±.05	.657±.06	.748±.04			.686±.04	
10	0			.649±.05	.694±.05	.691±.05	.645±.05	.678	.609±.05	
	1	.866±.04	.816±.05	.783±.05	.808±.04	.841±.03	.705±.06	.819±.03	.896±.02	.875±.04
	2			.809±.05	.857±.04	.792±.04	.853±.03	.610±.04	.707±.03	
13	0				.768±.05	.722±.05	.669±.05			
	1	.934±.03	.952±.02	.947±.02	.865±.03	.866±.03	.887±.02	.937±.02	.947±.02	.854±.05
	2				.833±.03	.903±.02	.906±.02			
15	0				.782±.04		.737±.05			
	1	.969±.02	.981±.02	.958±.02	.921±.03	.953±.02	.929±.02	.872±.05	.979±.02	.940±.02
	2				.868±.04		.915±.03			
17	0				.820±.04		.763±.05			
	1	.983±.02	.987±.01	.990±.01	.916±.03	.941±.03	.912±.04	.99	.99	.99
	2				.897±.04		.970±.02			

Table I-b

Optimal Strategies

λ = 0.1 db-hrs.; No passive sensors

U_{MAX}	3.	5.	7.	10.	13.	15.	17.
U_{OPT}	3.	5.	7.	10.	13.	15.	15.
CC_{OPT}	2	1	0	2	1	1	1
$IZIG_{OPT}$	1	3	1	1	2	2	2
$WEIGHT_{OPT}$	1.	.5	1.	1.	.5	.5	.5
Pr(No detection)	.295	.511	.621	.809	.865	.921	.921

Table II-a

No passive sonars

Detection rate: 0.1 db - hrs.

Sub U	CC	1 - ZIGZAG			2 - 'RANDOM'			3 - STRAIGHT LINE		
Destr IZIG / W		5	.7	1.	.5	.7	1.	.5	.7	1.
3	0		.049	.01						
	1	.318	.066	.01	.194	.230	.147	.138	.020	.206
	2		.081	.01						
5	0			.023	.163	.167		.034	.064	
	1	.438	.356	.118	.130	.124	.283	.183	.148	.430
	2			.033	.037	.204		.157	.054	
7	0			.253	.284	.343			.090	
	1	.597	.434	.265	.050	.244	.372	.476	.248	.641
	2			.160	.354	.374			.254	
10	0			.305	.441	.378	.320	.289	.193	
	1	.709	.610	.582	.553	.615	.447	.514	.630	.711
	2			.606	.660	.428	.587	.143	.234	
13	0				.490	.448	.388			
	1	.859	.876	.860	.677	.606	.676	.785	.878	.783
	2				.670	.699	.690			
15	0				.544		.470			
	1	.896	.917	.881	.815	.847	.825	.813	.933	.865
	2				.731		.779			
17	0				.600		.509			
	1	.943	.926	.943	.843	.890	.825	.988	.99	.987
	2				.769		.890			

Table II-b

Optimal Strategies

$\lambda = 1\ 0$ db-hrs.; No passive sensors

U_{MAX}	3.	5.	7.	10.	13.	15.	17.
U_{OPT}	3.	5.	7.	10.	13.	15.	17.
CC_{OPT}	1	1	2	1	1	1	1
$IZIG_{OPT}$	1	1	3	3	2	3	2
$WEIGHT_{OPT}$	1.	1.	.7	.5	.5	.5	.5
Pr(No detection)	0.10	.118	.254	.514	.677	.813	.843

Table III-a

Four passive sonars per active sonar

Detection rate: 0.1 db - hrs.

Destr IZIG		1 - ZIGZAG			2 - RANDOM			3 - STRAIGHT LINE		
Sub U	W / CC	5	.7	1.	.5	.7	1.	.5	.7	1.
3	0		.217	.155						
	1	.273	.199	.152	.320	.345	.333	.276	.226	.269
	2		.173	.178						
5	0			.188	.270	.290		.201	.205	
	1	.387	.382	.313	.315	.296	.382	.288	.303	.322
	2			.206	.211	.347		.246	.238	
7	0			.334	.274	.285			.287	
	1	.393	.338	.289	.197	.304	.312	.318	.291	.379
	2			.206	.269	.334			.244	
10	0			.277	.288	.287	.263	.290	.260	
	1	.322	.291	.279	.283	.281	.242	.318	.304	.315
	2			.187	.249	.251	.239	.135	.189	
13	0				.225	.210	.197			
	1	.208	.206	.196	.226	.180	.214	.193	.190	.132
	2				.118	.136	.141			
15	0				.164		.160			
	1	.138	.142	.140	.134	.140	.124	.098	.160	.082
	2				.071		.049			
17	0				.120		.112			
	1	0.99	.082	.067	.075	.075	.066	.011	.043	.012
	2				.039		.040			

Table III-b

Optimal Strategies

λ = 0.1 db-hrs.; 4 passive sonars per active sonar

U_{MAX}	3.	5.	7.	10.	13.	15.	17.
U_{OPT}	3.	5.	5.	10.	10.	10.	10.
CC_{OPT}	2	1	1	1	1	1	1
$IZIG_{OPT}$	1	3	3	3	3	3	3
$WEIGHT_{OPT}$	1.	.5	.7	.7	.7	.7	.7
Pr(No detection)	.170	.288	.303	.304	.304	.304	.304

Table IV-a

Four passive sonars per active sonar

Detection rate: 1.0 db - hrs.

Destr IZIG / W / Sub U	CC	1 - ZIGZAG 5	1 - ZIGZAG .7	1 - ZIGZAG 1.	2 - 'RANDOM' .5	2 - 'RANDOM' .7	2 - 'RANDOM' 1.	3 - STRAIGHT LINE .5	3 - STRAIGHT LINE .7	3 - STRAIGHT LINE 1.
3	0		.027	.01						
	1	.209	.043	.01	.123	.147	.093	.085	.012	.126
	2		.049	.01						
5	0			.013	.097	.099		.020	.038	
	1	.257	.208	.069	.075	.072	.160	.103	.083	.236
	2			.017	.019	.104		.078	.024	
7	0			.137	.150	.177			.048	
	1	.293	.211	.126	.024	.113	.173	.213	.117	.305
	2			.062	.145	.167			.090	
10	0			.130	.183	.157	.131	.124	.082	
	1	.264	.210	.207	.194	.206	.153	.200	.214	.256
	2			.140	.192	.136	.164	.032	.063	
13	0				.144	.130	.114			
	1	.192	.190	.178	.177	.126	.163	.162	.166	.121
	2				.089	.105	.108			
15	0				.114		.102			
	1	.127	.152	.129	.119	.125	.110	.092	.152	.075
	2				.060		.042			
17	0				.088		.075			
	1	.094	.077	.064	.069	.071	.060	.049	.01	.051
	2			.160		.182				

Table IV-b

Optimal Strategies

λ = 1.0 db-hrs., 4 passive sensors per active sensor

U_{MAX}	3.	5.	7.	10.	13.	15.	17.
U_{OPT}	3.	5.	7.	7.	7.	7.	17.
CC_{OPT}	1	1	1	1	1	1	2
$IZIG_{OPT}$	1	1	3	2	2	2	2
$WEIGHT_{OPT}$	1.	1.	.7	1.	1.	1.	1.
Pr(No detection)	.010	.069	.117	.173	.173	.173	.182

Table V-a

No passive sensors

Cookie cutter detection model

Destr IZIG Sub U	CC W	1 - ZIGZAG 5	.7	1.	2 - 'RANDOM' .5	.7	1.	3 - STRAIGHT LINE .5	.7	1.
3	0		.0	.0						
	1	.210	.030	.0	.140	.100	.050	.0	.0	.16
	2		.040	.0						
5	0			.0	.110	.110		.0	.030	
	1	.390	.320	.0	.0	.030	.050	.050	.010	.390
	2			.0	.0	.0		.0	.010	
7	0			.090	.200	.320			.020	.
	1	.550	.350	.110	.040	.140	.230	.213	.130	.620
	2			.030	.300	.170			.0	
10	0			.250	.340	.260	.210	.170	.150	
	1	.630	.440	.440	.420	.560	.350	.380	.500	.630
	2			.420	.630	.340	.410	.020	.100	
13	0				.430	.380	.330			
	1	.810	.850	.790	.600	.540	.630	.640	.740	.760
	2				.630	.680	.650			
15	0				.460		.390			
	1	.840	.850	.800	.750	.700	.640	.810	850	.840
	2				.620		.710			
17	0				.520		.450			
	1	.890	.860	.840	.840	.860	.780	.850	.980	.970
	2				.720		.880			

Table V-b

Optimal Strategies

Cookie-cutter detection; no passive sensors

U_{MAX}	3.	5.	7.	10.	13.	15.	17.
U_{OPT}	3.	5.	7.	10.	13.	15.	17.
CC_{OPT}	-	-	1	1	2	1	1
$IZIG_{OPT}$	1	1	1	3	2	2	2
$WEIGHT_{OPT}$	1	1	1	.5	.5	.7	.5
Pr(No detection)	.000	.000	.110	.380	.630	.700	.840

Table VI

Case 1

U = 10

CC = 1

IZIG = 2

WEIGHT = .7

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 0	r = 4
DSUB =				
0	$.758 \pm .06$	.325	.515	.220
5	$.845 \pm .04$	.361	.530	.227
10	$.779 \pm .06$	.313	.512	.206
20	$.841 \pm .04$	.281	.615	.206
25	$.786 \pm .05$	.277	.446	.157

Case 2

U = 5

CC = 1

IZIG = 3

WEIGHT = .5

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 0	r = 4
DSUB =				
0	$.419 \pm .06$	.261	.130	.081
5	$.500 \pm .06$	.307	.158	.097
10	$.537 \pm .06$	.318	.187	.111
20	$.511 \pm .06$	.288	.183	.103
25	$.385 \pm .06$	.213	.079	.044

Table VII

Case 1

U = 10

CC = 1

IZIG = 2

WEIGHT = .7

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 1.0	r = 4
α =				
1.0	.797$\pm$ .05	.245	.468	.144
.9	.847$\pm$ .05	.303	.630	.226
.8	.845$\pm$ .05	.303	.610	.219
.7	.848$\pm$ .04	.282	.615	.205
.5	.841$\pm$ .04	.281	.615	.206
.25	.836$\pm$ .05	.280	.615	.206

Case 2

U = 5

CC = 1

IZIG = 3

WEIGHT = .5

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 0	r = 4
α =				
1.0	.597$\pm$ .06	.326	.202	.110
.9	.534$\pm$ .06	.295	.188	.104
.7	.521$\pm$ .06	.294	.174	.098
.5	.511$\pm$ .06	.288	.183	.103
.25	.484$\pm$ .06	.272	.161	.091

Table VIII

Case 1

U = 10

CC = 1

IZIG = 2

WEIGHT = .7

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 0	r = 4
GNGFAC= 1.5	.805 ± .05	.285	.555	.197
1.0	.818 ± .06	.267	.626	.206
.75	.841 ± .04	.281	.615	.206
.50	.740 ± .06	.288	.474	.185
.25	.722 ± .06	.264	.486	.178

Case 2

U = 5

CC = 1

IZIG = 3

WEIGHT = .5

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 0	r = 4
GNGFAC =				
1.5	.493 ± .06	.256	.180	.093
1.0	.610 ± .08	.350	.395	.227
.75	.511 ± .06	.288	.183	.103
.50	.456 ± .06	.264	.123	.071
.25	.354 ± .05	.210	.028	.016

Table IX

U = 10
CC = 1
IZIG = 2
WEIGHT = .7

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 0	r = 4
BARDEP=				
10.	.821 ± .05	.311	.563	.213
7.5	.823 ± .04	.306	.543	.202
5	.841 ± .04	.281	.615	.206
2.5	.765 ± .07	.280	.657	.240
2	.776 ± .06	.313	.555	.224

Table X

Case 1

U = 10
CC = 1
IZIG = 2
WEIGHT = .7

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 0	r = 4
GAPSIZE=				
0.	.816 $\pm$ .05	.244	.566	.169
10.	.841 $\pm$ .04	.281	615	.206
20.	.693 $\pm$ .06	.304	.432	.190
30.	.699 $\pm$ .07	.361	.448	.231

Case 2

U = 5
CC = 1
IZIG = 3
WEIGHT = .5

	λ = 0.1 db - hrs.		λ = 1.0 db - hrs.	
	r = 0	r = 4	r = 0	r = 4
GAPSIZE=				
0.	.499 $\pm$ .06	.203	.157	.071
10.	.511 $\pm$ .06	.288	.183	.103
20.	.403 $\pm$ .08	.261	.265	.172
30.	.434 $\pm$ .08	.370	.341	.236

6. References

[1] Theory of Search II, Bernard O. Koopman, OPERATIONS RESEARCH, October 1956.

[2] Optimum Random Search Procedures for Detecting Evasive Targets, W. J. Fischer, February, 1970, Ph.D. Dissertation, School of Engineering and Applied Science, the George Washington University.

Appendix B-I. Subroutine WAIT

This subroutine determines whether the submarine can safely proceed North; if not, it determines the length of time, denoted by symbol AA, that it is necessary to "Wait" (defined below) before it is safe to proceed, assuming the destroyer maintains its present course. If the destroyer changes course during this "Wait" period, the situation is re-examined. The purpose of this subroutine is to advance the time of the next examination of the situation as far as possible; it is not necessary to consider events in the interim. This simulation thus differs from the conventional simulation procedure in which time proceeds in regular intervals of (say) .1 hours.

Two procedures are available during the "Wait" period.

(1) Only one destroyer has been detected

We calculate the required Wait, on the basis that during the Wait period the submarine proceeds either to the East or to the West at speed U. During this period, a CPA (closest point of approach) is experienced with the destroyer. The direction - East or West - yielding the largest CPA is chosen. In the event that the two CPA's are equal (this happens when the destroyers track is precisely in the E-W direction), the direction leading away from the destroyer is chosen.

(2) Two (or more) destroyers have been detected

The submarine is required to lie d.i.w. (dead in water) during the Wait period. The motivation for this is that, in most situations, a movement to the East or West would be of no advantage when opposing two destroyers, and would increase the chance of detection through passive means.

<u>Analysis</u>

The solid line ———— indicates destroyer motion relative to the submarine during the WAIT period. The submarine's absolute velocity during this time is

(i) $\underset{\rightarrow}{U}$ or $\underset{\leftarrow}{U}$ (two cases)

(ii) zero. (d.i.w.)

according to the description in the previous section.

The dotted line ----------- indicates destroyer relative motion after the Submarine resumes its Northerly motion, and results in a CPA of precisely RMINP (provided destroyer motion does not change).

All co-ordinates are destroyer positions relative to the submarine. (XR1,YR2) is the initial relative position of the destroyer (XR2,YR2) is an arbitrary position along this relative track of the destroyer, used to define the track conveniently. (XINT,YINT) corresponds to the time at which the submarine resumes its Northerly course.

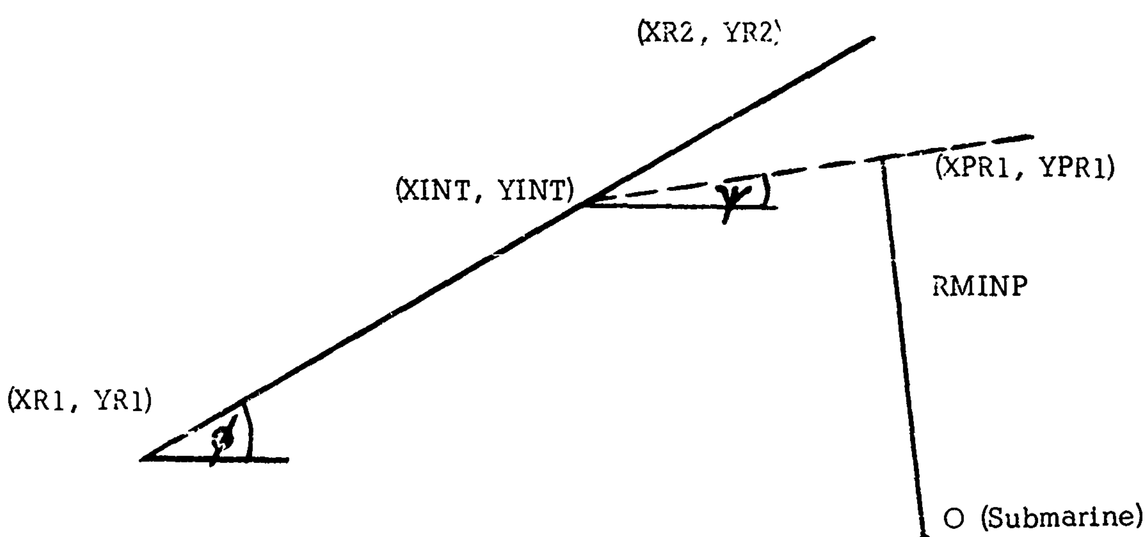


Figure B-3. Geometry for WAIT procedure

(XINT , YINT) are obtained as the solution for (X,Y) of

$$Y - YR1 = \tan\phi\,(X - XR1)$$

$$(Y \underset{(+)}{-} RMIN \cdot \cos\Psi) = (X \underset{(-)}{+} RMIN \sin\Psi)\tan\Psi$$

or (after some manipulation)

$$(\tan\phi - \tan\Psi)\ X = \underset{(-)}{+}\ \frac{RMIN}{\cos\Psi} + (\tan\phi \cdot XR1 - YR1)\ .$$

Note that, taking RMIN to be + or -, we obtain two formal solutions for (XINT , YINT). One of these will correspond to the situation where the submarine waits for the destroyer to approach, and then heads North just in time to achieve a CPA of RMINP. By leaving earlier, the submarine could achieve a larger CPA.

This is clearly unrealistic, when the submarine is opposing a single (detected) destroyer. In other words, the submarine would not wait the indicated period, but would go North immediately.

However, the two cases have been included to allow for their possible use in situations where (say) the submarine is attempting to penetrate between two destroyers. No use has been made of this in the current work. Instead, a simpler procedure is used to deal with two or more destroyers, as follows.

In our application, if both solutions are feasible, the subroutine would (already) have indicated a wait of zero. The program would then examine any other detected destroyers to see if a wait was desirable in their case. The only case of interest to us is where only one of the solutions is feasible, and the program determines which this is. For completeness, if both solutions are feasible, the earlier one (i.e. the one with the earlier time corresponding to (XINT, YINT)) is used. A tentative decision to "go" (proceed Northwards immediately is made if at least one of the following holds:

(i) The resulting CPA exceeds RMINP

(ii) The resulting CPA exceeds GNGFAC times the (best) CPA obtained by waiting for a favorable opportunity to transit (this latter CPA will occur during the WAIT phase).

The motivation for this is that a resulting CPA no worse than GNGFAC of the CPA obtained by waiting is worth risking, because of assumed time constraints on transit, and increased risk of detection while lying off the barrier.

In the event that a tentative decision to "go" is made, we examine also what will happen if the (best) Wait procedure were used. If the resulting CPA is <u>less than</u> RDFAC of RMINP, we clearly should not Wait under any circumstances (for example, on account of another destroyer). In this case, AA is set equal to -1 as a flag, and further examination of the other destroyers is abandoned.

If, on the other hand, there is no such urgency, or if the tentative decision is to wait, we proceed to examine the other detected destroyers. If no urgent "GO" indication is found, the final recommended WAIT period is the <u>maximum</u> of the WAIT periods for the detected destroyers.

Appendix B-II. Calculation of Detection Probability

(Describes INTEG Subroutine, and parts of GRIND subroutine)

We assume that the signal excess, in db, of the returned ping over background noise may be expressed as

$$E = \text{Constant} - 40 \log_{10} r$$

where r is the range in n. mi.

Let us introduce the symbol RMIN, corresponding to the range at which E becomes zero. Then it is convenient to define

$$E = \begin{cases} 40 \log_{10} (r/\text{RMIN}), & r < \text{RMIN} \\ 0 & , \quad r > \text{RMIN} \end{cases}$$

The probability of survival for a given submarine transit, writing E, r as functions of t, will be

$$\text{(1)} \quad \exp\left[-\int_{t_1}^{t_2} \lambda\, E(t)\, dt\right] = \exp\left[-\int_{t_1}^{t_2} \lambda\, 40 \log_{10} (r(t)/\text{RMIN})\, dt\right]$$

$$= \exp\left[-20\,(.43429)\,\lambda \left\{ \int_{t_1}^{t_2} \log_e r^2(t)\, dt - (\log_e \text{RMIN}^2)(t_2 - t_1) \right\}\right]$$

where λ is a constant

t_1, t_2 are the first and last times at which $r = \text{RMIN}$.

The last expression (1) is calculated in the subroutine GRIND, using 13 different values for λ.

The expression $(t_2 - t_1)$ is simply the accumulated time during which the range is less than RMIN, and this is accumulated in the subroutine RANGE.

The expression

$$\int \log_e r^2(t) \cdot dt$$

is to be accumulated only during the same period, ensured by the subroutine RANGE again, which calls the subroutine INTEG for the computation of each portion of the integral. It is convenient to calculate the integral for intervals of time during which the motion of the destroyer relative to the submarine is straight line.

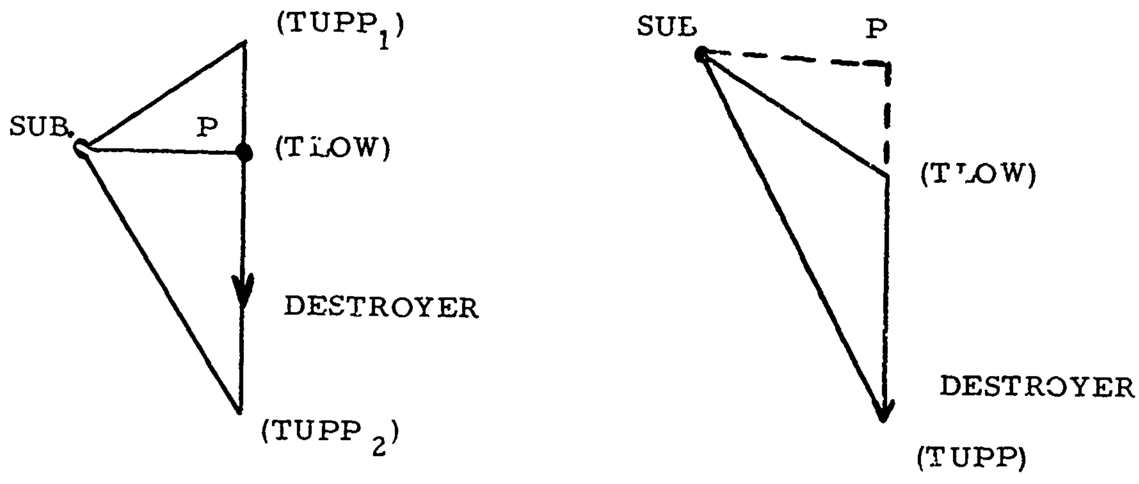
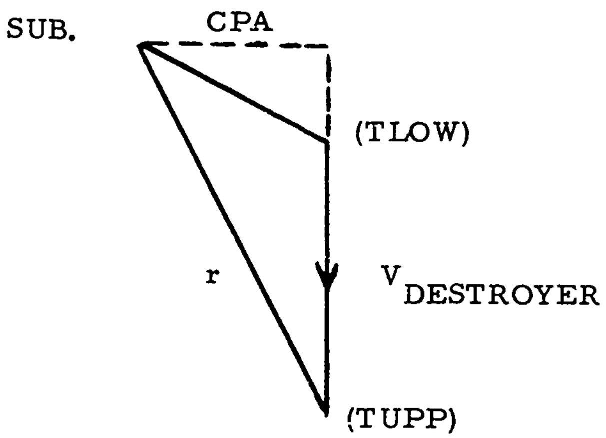


Figure B-9. Straight line segments of relative motion

Various situations can arise. and the "book-keeping" is handled by the subroutine RANGE. It is clear that INTEG need only calculate the integral in the following standard situation.



Fiqure B-10. Standard Situation for INTEG

Now $r^2 = a + b\ t^2$, where $a = CPA^2$, $b = v^2$

Thus $\int \log_e (r^2)\, dt = \int \log_e (a + b\ t^2)\, dt$

$$= t \log (a + b\ t^2) - 2t + 2(a/b)^{1/2} \tan^{-1}[(b/a)^{1/2} t]$$

This expression is defined in INTEG as function FN(T), and T is replaced by TUPP and TLOW in turn.

Appendix B-III. Contribution of Passive Sensors to Detection Probability (Subroutine GRIND)

Assume that the distance at which the submarine can be heard by a passive detector is given by

$$\overline{R} = 2 \cdot 10^{(.05\,U)} \text{ n. mi.}$$

The average duration of the submarine's transit (time counting only while the submarine is in motion) is called (BB/FLII) hours. The area swept out by a circle radius $\overline{R}$ with center the submarine, ignoring overlapping sections, is given by

$$2\,\overline{R}\,(BB/FLII)\,U$$

where U = speed of the submarine (knots).

Assume there are n sensors located randomly in an area

$$4 \cdot (BARDEP + RMIN) \cdot (BARWID + GAP)$$

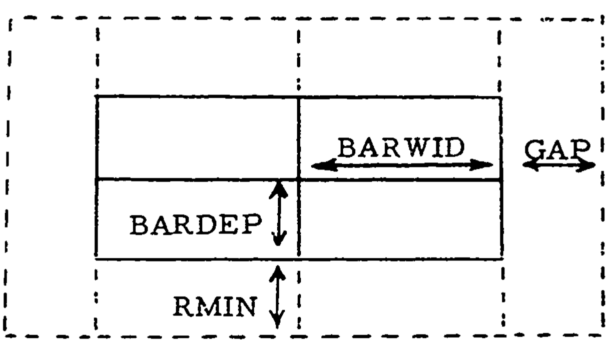


Figure B-11. Region containing 1 active, n passive, sensors

Then the probability of no detection by passive sensors is given by

$$[\exp\{-2\ \overline{R}\ (BB/FLII)\ U/AREA\}]^n$$

This expression is computed for various values of n, which is the ratio of passive, active sensors (assume each destroyer is an active sensor). These survival probabilities are applied in turn to the survival probability relating to detection by active sensors.

It is assumed that no interference is caused to passive detection by active detection procedures.

C. THE BARRIER PENETRATION PROBLEM

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C. THE BARRIER PENETRATION PROBLEM

1. General Discussion

The timing of an attempted penetration through a screen or barrier is an important tactic for a submarine. The exposure risk in waiting makes an early penetration desirable while the additional information and possibly more favorable positioning available by delaying makes a later penetration desirable. In addition, there will generally be greater utility in penetrating during some periods than during others.

When destroyers comprise the screen, they will generally employ active sonar. This allows the submarine to acquire information about the probability of detection for the various courses of action available to it. Furthermore, this information is constantly being revised until the submarine dives and actually begins the transit. In this case, the submarine is faced with the decision of when to stop waiting and to initiate an action which we will regard here as irreversible. A problem of this sort is called a stopping rule problem although in our case it might have been more appropriate if it were termed a "starting" rule problem. An analysis of the submarine penetration problem where it is constantly receiving information from active sonar destroyers is contained in the next section.

In some instances, the submarine cannot regularly obtain new data about the screen. This can arise from such factors as silent defenders (destroyers, submarine, aircraft), inability to hear through self-noise, and separations beyond the range at which the submarine can hear any sonar. To model this situation we have assumed that the submarine may make an initial projection of the probable future location of defensive craft (using a uniform distribution if no information at all is available). Since no additional information is received the optimal waiting period and subsequent movement through the screen can be determined in advance. In the third section we present a model of this situation which uses a discrete control theory approach and which can be solved by dynamic programming.

It is our feeling that these models are fairly realistic representations of present submarine performance and are suitable for use in models of submarine-barrier as screen encounters. However, there are situations where submarine strategy could probably be improved by application of these methods. This is particularly true of the first model described below.

2. Stopping Rule Analysis of the Barrier Penetration Problem

a. Introduction

We assume that a submarine is attempting to penetrate, without being detected, barrier or screen patrolled by one or more destroyers. Since the destroyers employ active sonar equipment, the submarine is able to detect the presence of the destroyers before the destroyers are able to detect the submarine. The problem faced by the submarine is whether to continue across the barrier or whether to wait for more fortuitous positions of the destroyers along the barrier. If it waits for the period, the submarine may reposition itself to be better situated at the next period or to avoid a potential threat during the current period. In this report we formulate the barrier penetration problem as a stopping rule problem for the finite horizon case and develop the dynamic programming recursion which computes the optimal policy for the submarine. We then extend these results to allow an infinite planning horizon.

b. Definitions

Let $t_1, t_2, \ldots, t_i, \ldots$ be a set of times such that either:

(a) The submarine first detects one or more destroyers at t_i;

(b) The submarine detects one or more destroyer course changes at t_i; or

(c) The submarine no longer detects one or more destroyers after t_i.

Let $P(t_i)$ be the probability that the submarine will be detected after t_i if it leaves at its best time within the interval $[t_i, t_{i+1})$ and with its best course.

Let $A(t_i)$ be the probability that the submarine will be detected during the interval $[t_i, t_{i+1})$ if it does not initiate transit.

We assume that the destroyers employ random search strategies (e.g., the destroyers follow straight line trajectories for random lengths of time, followed by random course changes). Consequently, $P(t_i)$ and $Q(t_i)$ are random variables for $i = 1, 2, \ldots$ However, at t_i, we assume that the submarine is able to make reasonable estimates for $P(t_i)$ and $Q(t_i)$ and, on the basis of these estimates, determine whether to continue across the barrier during $[t_i, t_{i+1})$ or whether to wait for the entire interval. We assume that the random variables $(P(t_i), Q(t_i))$ and $(P(t_i), Q(t_j))$ are statistically independent if $i \neq j$, but we do allow $P(t_i)$ and $Q(t_i)$ to have a joint density function. Let $f[P(t_i), Q(t_i)]$ be the joint probability density function for the random variables for $i = 1, 2, \ldots$

Let $\rho_i(\cdot)$ be a Borel measurable* policy defined as follows: if

$$\rho_i [P(t_i), Q(t_i)] = 1$$

then the submarine leaves on its best course and at its best time during the interval $[t_i, t_{i+1})$; and if

----

* The class of Borel measurable functions is defined in standard texts such as Refs. 1 and 2.

$$\rho_i\,[P\,(t_i),\ Q\,(t_i)] = 0\,,$$

then the submarine waits at least until t_{i+1}. The policy $\rho_i\,(\cdot)$, $i = 1, 2, \ldots,$ defines the submarine strategy to be used. In the next two sections we will characterize the form of the optimal submarine policy.

It is convenient to define the functions

$$H\,[\rho_i\,(\cdot)] = \iint \rho_i\,(P,\ Q) \cdot P \cdot f\,(P,\ Q)\, dPdQ$$

and

$$G\,[\rho_i\,(\cdot),\ R_{i+1}]$$

$$= \iint [1 - \rho_i\,(P,\ Q)]\,[R_{i+1} \cdot (1-Q)+Q]\, f(P,Q)dP\,dQ\,.$$

c. Finite Horizon Case

In the finite horizon case, the submarine must start crossing the barrier within the inteval $[t_1,\ t_T)$. The function $\rho_i\,[P\,(t_i), Q(t_i)]$ is said to be an admissible stopping rule policy if:

(a) $\rho_i\,[P(t_i),\ Q(t_i)]$ is a Borel measurable function of $P(t_i)$ and $Q(t_i)$;

(b) $\rho_i\,[P(t_i),\ Q(t_i)] \in \{\,0,\ 1\,\}$ for $0 \leq P(t_i) \leq 1$ and $0 \leq Q(t_i) \leq 1$; and

(c) if $i = T - 1$, then

$$\rho_i\,[P(t_i),\ Q(t_i)] = 1 \qquad (1)$$

The last condition insures that the submarine will start crossing the barrier within the time interval $[t_1, t_T)$. The submarine strategy over the entire planning horizon is defined by the functions $\rho_1(\cdot), \rho_2(\cdot), \ldots \rho_{T-1}(\cdot)$. It is convenient to use the convention that

$$\overline{\rho}_i(\cdot) = [\rho_i(\cdot), \rho_{i+1}(\cdot), \ldots, \rho_{T-1}(\cdot)].$$

The function $\overline{\rho}_i(\cdot)$ is aid to be <u>admissible</u> if and only if the functions $\rho_i(\cdot), \rho_{i+1}(\cdot), \ldots, \rho_{T-1}(\cdot)$ are admissible policies.

Let $R_i[\overline{\rho}_i(\cdot)]$ be the expected probability of detection of the submarine after t_i, given that the submarine waiting during $[t_1, t_i)$ and the admissible policy $\overline{\rho}(\cdot)$ is used. Here $R_i[\overline{\rho}_i(\cdot)]$ is a complex function of $\overline{\rho}_i(\cdot)$ and $f(\cdot)$.

If there exists a policy $\overline{\rho}_i^*(\cdot)$ such that

$$R_i[\overline{\rho}_i^*(\cdot)] \leq R_i[\overline{\rho}_i(\cdot)]$$

for all admissible policies $\overline{\rho}_i(\cdot)$, then $\overline{\rho}_i^*(\cdot)$ is said to be the <u>optimal i^{th} stage stopping rule policy</u>. We will prove that $\overline{\rho}_i^*(\cdot)$ exists for $i = 1, 2, \ldots, T-1$ and that it satisfies

$$\overline{\rho}_i^*(\cdot) = [\rho_i^*(\cdot), \overline{\rho}_{i+1}(\cdot)]$$

for some function $\rho_i^*(\cdot)$.

THEOREM 1: The optimal i^{th} stage stopping rule policy $\bar{\rho}_i^*(\cdot)$ exists and is defined as

$$\bar{\rho}_i^*(\cdot) = [\rho_i^*(\cdot), \rho_{i+1}^*(\cdot), \ldots, \rho_{T-1}^*(\cdot)] \tag{2}$$

for $i = 1, \ldots, T-1$, where

$$\rho_i^*[P(t_i), Q(t_i)] = \begin{cases} 1 & \text{if } P(t_i) \leq R_{i+1}^*[1 - Q(t_i)] + Q(t_i) \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

$$R_i^* = R[\bar{\rho}_i^*(\cdot)] \quad , \tag{4}$$

and R_i^* satisfies the recursion

$$R_i^* = H[\rho_i^*(\cdot)] + G[\rho_i^*(\cdot), R_{i+1}^*] \tag{5}$$

for $i = 1, \ldots, T-1$, subject to the terminal condition

$$R_T^* = 1 \quad . \tag{6}$$

PROOF: This result is proved by induction. Consider first $i = T-1$. By Eqn. 1, $\rho^*_{T-1}(\cdot)$ exists and is defined as

$$\rho^*_{T-1}[P(t_{T-1}),Q(t_{T-1})] = 1$$

for all $P(t_{T-1})$ and $Q(t_{T-1})$. Since $R^*_T = 1$, it is easy to verify that $\rho^*_{T-1}(\cdot)$, R^*_{T-1}, and R^*_T do satisfy Eqns. (3)-(5).

Next assume that the theorem is true for $i+1, i+2, \ldots, T-1$, and we will prove the result for i. It follows from the definitions of $R_i[\bar{\rho}_i(\cdot)]$, $H[\rho_i(\cdot)]$, $G[\rho_i(\cdot), R_{i+1}[\bar{\rho}_{i+1}(\cdot)]]$ and the independence of the random variables $(P(t_i), Q(t_i))$ and $(P(t_{i+1}), Q(t_{i+1}))$ that $R_i[\bar{\rho}_i(\cdot)]$ satisfies the recursion

$$R_i[\bar{\rho}_i(\cdot)] = H[\rho_i(\cdot)] + G[\rho_i(\cdot), R_{i+1}[\bar{\rho}_{i+1}(\cdot)]] \tag{7}$$

for any admissible policy $\bar{\rho}_i(\cdot) = [\rho_i(\cdot), \bar{\rho}_{i+1}(\cdot)]$. By the definition of G and the induction assumption,

$$R_i[\bar{\rho}_i(\cdot)] \geq H[\rho_i(\cdot)] + G[\rho_i(\cdot), R^*_{i+1}] \quad . \tag{8}$$

By the definitions of $H[\rho_i(\cdot)]$ and $G[\rho_i(\cdot), R^*_{i+1}]$,

$$H[\rho_i(\cdot)] + G[\rho_i(\cdot), R^*_{i+1}] \tag{9}$$

$$= \iint M[P, Q, R^*_{i+1}, \rho_i(\cdot)]\ f(P,Q)\ dP\ dQ \quad ,$$

where

$$M[P,Q,R^*_{i+1},\rho_i(\cdot)] = \begin{cases} P & \text{for } \rho_i(P,Q) = 1 \\ Q + R^*_{i+1}(1-Q) & \rho_i(P,Q) = 0 \end{cases}$$

If we define

$$\rho^*_i(P,Q) = \begin{cases} 1 & \text{for } P \leq Q + R^*_{i+1} \cdot (1-Q) \\ 0 & \text{otherwise}, \end{cases} \tag{10}$$

then

$$M[P,Q,R^*_{i+1},\rho^*_{i+1},\rho^*_i(\cdot)] \leq M[P,Q,R^*_{i+1},\rho_i(\cdot)]$$

for any values of P, Q, and R^*_{i+1} and for any admissible function $\rho_i(\cdot)$. After integrating the above inequality, we have

$$\iint M[P,Q,R^*_{i+1},\rho_i(\cdot)]\ f(P,Q)\ dP\ dQ$$

$$\leq \iint M[P,Q,R^*_{i+1},\rho_i(\cdot)]\, f(P,Q)\, dP\, dQ \quad ,$$

which along with Eqns. (8)-(9) implies that

$$R_i[\overline{\rho}_i(\cdot)] \geq H[\rho_i^*(\cdot)] \quad G[\rho_i^*(\cdot),R^*_{i+1}] \quad . \tag{11}$$

If we define

$$\overline{\rho}_i^*(\cdot) = [\rho_i^*(\cdot),\overline{\rho}_{i+1}(\cdot)] \quad ,$$

then by Eqn. (7)

$$R_i[\overline{\rho}_i^*(\cdot)] = H[\rho_i^*(\cdot)] + G[\rho_i^*(\cdot),R^*_{i+1}] \tag{12}$$

Thus by Eqns. (11)-(12),

$$R_i[\overline{\rho}_i^*(\cdot)] \leq R_i[\overline{\rho}_i(\cdot)]$$

for all admissible policies $\overline{\rho}_i(\cdot)$. Eqn (10) implies Eqn. (3), and Eqn. (12) implies Eqns. (4)-(5), which completes the proof of the theorem.

d. Infinite Horizon Case

In this section, we extend the results of the previous section to allow an infinite planning horizon. This is appropriate when there is no required time for initiating a penetration attempt and the risk in waiting is the compelling factor for an early transit. Let $R_i^*(t)$ be the minimum probability of detection of the submarine after t_i, given that the submarine waited during $[t_1, t_i)$ and that the submarine must start crossing the barrier within the interval $[t_i, t_T)$.

THEOREM 2: For any fixed i, the limit

$$R^* = \lim_{T \to \infty} R_i^*(T) \tag{13}$$

exists.

PROOF: It is a simple exercise to show that

$$R_i^*(T+1) \leq R_i^*(T).$$

Since $R_i^*(T)$ is a probability, the sequence $\{R_i^*(i), R_i^*(i+1), \ldots\}$ is bounded from below by zero. Thus the theorem follows immediately from the Monontone Convergence Theorem (Theorem 12.1 in Ref. 3).

We next characterize the form of the limit R^*. If X is a random variable, then we represent the expectation of X as $E\{X\}$.

THEOREM 3: R^* satisfies

$$(14) \qquad E\{\min[P(t_i);Q(t_i)]\} \le R^* \le E\{P(t_i)\}$$

for $i = 1,2,\ldots$

PROOF: By Eqn. (10),

$$M[P,Q,R^*_{i+1}(T),\rho^*_i(\cdot)] = \min[P;Q + R^*_{i+1}(T)\cdot(1-Q)]$$

which is a continuous function of P,Q, and $R^*_{i+1}(T)$. Thus it follows from Theorem 23.9 in Ref. 3 that

$$\iint \min[P;Q + R^*_{i+1}(T)\cdot(1-Q)]\, f(P,Q)\, dP\, dQ$$

is a continuous function of $R^*_{i+1}(T)$. By Theorem 2 and an elementary property of continuous functions (Theorem 15.2 in Ref. 3),

$$\lim_{T\to\infty} \iint \min[P;Q + R^*_{i+1}(T)\cdot(1-Q)]\, f(P,Q)\, dP\, dQ$$

$$= \iint \min[P;Q + R^*\cdot(1-Q)]\, f(P,Q)\, dP\, dQ\ .$$

By Eqns. (9) and (12),

$$\lim_{T\to\infty} R_i^*(T) = \lim_{T\to\infty} \iint \min[P; Q+R_{i+1}^*(T) \cdot (1-Q)] \; f(P,Q) \, dP \, dQ$$

which implies that

$$R^* = \iint \min[P; Q+R^* \cdot (1-Q)] \, f(P,Q) \, dP dQ \;.$$

Since $\min[P; Q+R^*(1-Q)] \leq P$,

$$R^* \leq \iint P \, f(P,Q) \, dP dQ = E\{P\}$$

which is the right hand side of Eqn. (14). Also,

$$\min[P; Q+R^* (1-Q)] \geq \min[P;Q] \;,$$

and thus

$$R^* \geq \iint \min[P;Q] \; f(P,Q) \, dP \, dQ = E\{\min[P;Q]\}$$

which is the left hand side of Eqn. (14) and completes the proof of the theorem.

For density functions $f[P(t_i), Q(t_i)]$ that are encountered in practice, it is expected that

$$E\{P(t_i)\} < 1$$

and that

$$E\{\min [P(t_i)]\} > 0;$$

thus Theorem 3 implies that the limit R^* will be nontrivial (i.e. $0 < R^* < 1$).

e. Conclusions

Theorems 1-2 imply that the optimal stopping rule policy $\rho_i^*(\cdot)$ for an infinite planning horizon is in the following form:

$$\rho_i^* [P(t_i), Q(t_i)] = \begin{cases} 1 & \text{if } P(t_i) \le R^* \cdot [1 - Q(t_i)] + Q(t_i) \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where R^* is a constant. A method to determine R^* is the following: use simulation experiments to compute the submarine detection probability for several values of R^*, and then pick that value which results in the minimum detection probability.

Some comments on the use of the policy arising from Eqn. (15) are as follows:

(a) Although the destroyers may make random course changes as the times t_i, t_{i+1}, etc., the random variables $[P(t_i), Q(t_i)]$ and $[P(t_j), Q(t_j)]$, $i \neq j$, probably will not be stochastically independent, particularly if there are several destroyers.

(b) There may be different values of R^* for different positions of the submarine in the barrier; thus the empirical procedure for measuring R^* described above results, in some sense, in an average value for R^*.

(c) Since the destroyers use random search tactics, estimating $P(t_i)$ and $Q(t_i)$ may be quite difficult in practice. One approach is to assume that the submarine can start across the barrier only at times t_i, t_{i+1}, etc., and then estimate $P(t_i)$ and $Q(t_i)$ by assuming that the destroyers and the submarine travel indefinitely on straight line trajectories.

(d) The appropriate value for R^* depends upon the submarine's speed, the search tactics of the destroyers, and the dimensions of the barrier. After determining R^* for a number of combinations of different values for these parameters, it may be possible to develop a functional relationship between R^* and these variables.

3. Dynamic Programming Approach to the to the Barrier Penetration Problem

a. Introduction

This model arises when a submarine attempts to penetrate a finite barrier that is patrolled by one or more vessels and when the following assumptions are valid:

1. The submarine is able to estimate a probability density function for the locations of the vessels patrolling the barrier;

2. The probability that the submarine can be detected is a function of only the coordinates of the sonar sensor and the submarine (i. e., it is not a function of their velocities); and

3. The submarine is not allowed to bypass the barrier, but must attempt to penetrate it within T time periods.

The probability density function described in one may be simply a uniform distribution if no information is available. We will develop a dynamic programming algorithm which computes the submarine trajectory that minimizes the expected number of detections prior to penetration.

b. Model Formulation

The geometry of the model is illustrated in the following figure:

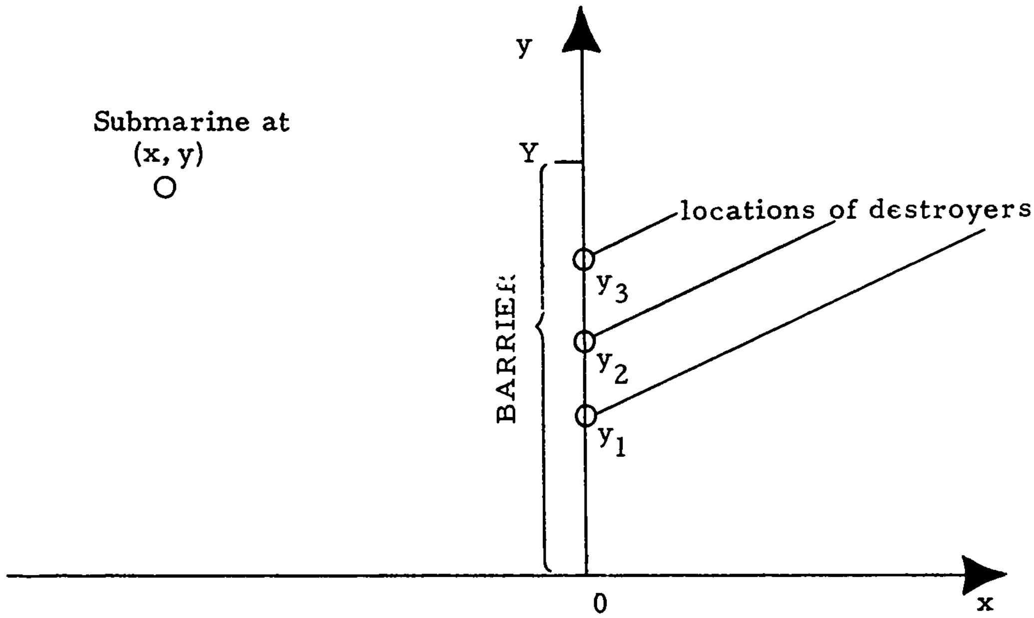

Figure C-1: Problem Geometry

The finite barrier is between (0, 0) and (0, Y); the coordinates of the submarine are (x, y); and the coordinates for Destroyer D_i are $(0, y_i)$. Define:

T = the maximum number of time periods available for the submarine to penetrate the barrier,

$P_i(y_i, t)$ = the probability density function for destroyer D_i to be at location $(0, y_i)$ in time period t,

$q(x, y, y_i)$ = the probability that a destroyer located at $(0, y_i)$ will detect the submarine located at (x, y) during a time period,

$\bar{i}$ = the total number of destroyers that are patrolling the barrier.

Estimates for the above parameters and functions are assumed to be known by the submarine.

We will use difference equations to describe the motion of the submarine. Let

$(x(t), y(t))$ = the coordinates of the submarine at the beginning of the t^{th} time period.

$(u(t), v(t))$ = the control variables affecting the submarine's trajectory.

The difference equations relating the trajectory to the control variables are

$$x(t+1) = x(t) + A\,u(t)$$
$$y(t+1) = y(t) + B\,v(t) \tag{1}$$

The control variables satisfy the constraints

$$u(t) \in \{ \underline{U}, \underline{U}+1, \ldots, -1, 0, +1, \ldots, \overline{U} \}$$
$$v(t) \in \{ \underline{V}, \underline{V}+1 \ \ldots, -1, 0, +1, \ldots, \overline{V} \} \tag{2}$$

and (3)

$$u(t)^2 + v(t)^2 \leq R^2$$

The initial coordinates of the submarine are

$$x(0) = x^0 \text{ and } y(0) = y^0 \tag{4}$$

We assume that A, B, $\underline{U}$, $\overline{U}$, $\underline{V}$, $\overline{V}$, x^0, and y^0 are integers and that (x^0/A) is also an integer.

An admissible trajectory $X(x^s, y^s, s, \tau)$ is defined to be a set of coordinates

$$X(x^s, y^s, s, \tau) = \{ (x(t), Y(t)): t = s, s+1, \ldots, \tau \}$$

which satsify Eqns. (1)-(3), the coordinates

$$x(s) = x^s \text{ and } y(s) = y^s, \tag{5}$$

the constraints

$$0 \leq y(t) \leq Y \text{ for } t = s, s+1, \ldots, \tau \tag{6}$$

and the terminal condition

$$x(\tau) = 0.$$

Define

$$h[x, y, t] = 1 - \prod_{i=1}^{\bar{i}} \int_{y_i=0}^{Y} (1 - q(x, y, y_i)) \rho_i (y_i, t) dy_i; \tag{7}$$

then the expected number of detections of the submarine when following the admissible trajectory $X(x^s, y^s, s, \tau)$ is

$$C[X(x^s, y^s, s, \tau)] = \sum_{t=s}^{\tau} h[x(t), y(t), t] \tag{8}$$

Let

F = the set of admissible trajectories $X(x^0, y^0, 0, \tau)$ such that $\tau \leq T$;

then an admissible trajectory $X^*(x^0, y^0, 0, \tau^*)$ is said to be <u>optimal</u> if

$$C[X^*(x^0, y\ , 0, \tau^*)] \leq C[X(x^0, y^0, 0, \tau)]$$

for all $X(x^0, y^0, 0, \tau) \in F$.

c. <u>Dynamic Programming Algorithm</u>

We now develop an algorithm based upon dynamic programming which computes the optimal admissible trajectory. Let

$f(x^s, y^s, s, \tau)$ = the minimum expected number of detections of a submarine travelling on an admissible trajectory from (x^s, y^s) at time period s to the barrier at time period τ.

This function satisfies the recursion

$$f(x^s, y^s, s, \tau) = h(x^s, y^s, s) \qquad = h(x^s, y^s, s) \tag{9}$$

$$+ \min \{ f(x^s + A\, u(s), y^s + B\, v(s), s+1, \tau)\},$$

where the minimization is over the set of values $(u(s), v(s))$ satisfying constraints (2)-(3). By definition

$$f(0, y^\tau, \tau, \tau) = 0 \,. \tag{10}$$

Thus by using Eqns. (9)-(10), $f(x^s, y^s, s, \tau)$ can be computed for all values (x^s, y^s, s) for which there exits an admissible trajectory $X(x^0, y^0, 0, \tau)$ such that $x(s) = x^s$ and $y(s) = y^s$. The trajectory $X^*(x^s, y^s, s, \tau)$ satisfying

$$f(x^s, y^s, s, \tau) = C[X^*(x^s, y^s, s, \tau)]$$

is also determined from these computations. And finally, the optimal admissible trajectory $X^*(x^0, y^0, 0, \tau^*)$ is found by solving the minimization problem:

$$C[X^*(x^0, y^0, 0, \tau^*)] = \min \{C[X^*(x^0, y^0, 0, \tau)]: \tau \leq T\}$$

4. Theoretical Analysis of Threshold Levels

a. Introduction

In section 2 we developed the stopping rule for use by a submarine which continually receives new information while it is waiting. As expressed in equation 15 for an infinite time horizon, the submarine should stop waiting and begin the transit if and only if

$$p(t_i) \leq R^* \cdot \left[1 - Q(t_i)\right] + Q(t_i). \tag{1}$$

The quantity $p(t_i)$ is the current estimate of the probability of detection if the submarine begins the transit in time t_i and $Q(t_i)$ is the current estimate of the probability of detection between t_i and t_{i+1} if the submarine waits.

The value of R^* is to be determined as a function of the geometry of the barrier, the speed of vessels, the destroyer search tactics, etc. It is possible to determine R^* by simulation as suggested in section 2; however, it is our objective in this section to explore theoretical procedures for obtaining this quantity.

b. Analysis

It will be convenient to introduce some new notation as follows,

x_i = the probability of a successful penetration if transit is begun in period t_i. (This equals $1 - P(t_i)$ in section 2.)

q = the probability that the submarine will survive between any two successive periods if transit is not begun. (This equals $1 - Q(t_i)$ where $Q(t_i)$ is independent of t_i.)

ρ = the threshold for initiating a transit.

The value of ρ is related to R^*. By substituting in equation (1) above we find that transit should be initiated if and only if:

$$1 - x_i \leq R^* q + (1-q) \tag{2}$$

or

$$x_i \geq q - R^* q = q(1-R^*) \tag{3}$$

Thus the threshold is related to R^* by the equality:

$$\rho = q(1-R^*) \tag{4}$$

In this section we will work with ρ instead of R^*.

The value of x_i obtained at each period t_i may be regarded as being drawn from some (unknown) distribution $F(x)$. At period t_i we have a history of x_i's obtained from the distribution. We shall assume that the differences $\Delta t_i = t_i - t_{i-1}$ are sufficiently large so that the x_i are statistically independent; we also may regard the Δt_i as fixed at some Δt to conform to our definition of q as a constant. From the x_i so obtained, we can determine a sample mean and variance at time i. Thus we define:

$$\bar{x}_i = \left[\sum_{j=1}^{i} x_j \right] / i \tag{5}$$

$$s_i = \left[\sum_{j=1}^{i} (x_j - \bar{x}_i)/(i - 1) \right]^{\frac{1}{2}} \tag{6}$$

With this information we can estimate F(x) at time t_i; we will call this distribution $F_i(x)$.

Since x_i must lie in the range [0, 1], the distributions must satisfy:

$$F(0) = 0 \qquad F_i(0) = 0 \tag{7a, 7b}$$

$$F(1) = 1 \qquad F_i(1) = 1 \tag{8a, 8b}$$

It will be convenient in the following text to use the distribution complements defined by:

$$\overline{F}(x) = 1 - F(x) \tag{9}$$

$$\overline{F_i}(x) = 1 - F_i(x) \tag{10}$$

Finally let us assume that the distributions are continuous and possess derivatives given by f(x) and $f_i(x)$, the latter being associated with the $F_i(x)$.

At period t_i we can form the current best estimate of ρ which we will call r_i. It will be a function of $F_i(x)$ and consequently a function of $\bar{x}_i$ and s_i.

We can also define the current estimate of the expected probability of success v_i as follows:

$$v_i = \int_{r_i}^{1} x \, d\, F(x) + F(r_i)\, q\, v_{i+1} \tag{11}$$

The first part of the expression is the probability of a successful transit if one is undertaken and the second part is the probability of waiting until period t_{i+1} times the probability of remaining undetected during Δt_{i+1} times the probability of success in period t_{i+1}. The true value of the expected probability of success v can be defined as follows:

$$v = \int_{\rho}^{1} x \, d\, F(x) + F(\rho)\, q\, v. \tag{12}$$

The first parts of equations (11) and (12) may be reformulated by integrating by parts. Thus we obtain:

$$\int_{r_i}^{1} x \, d\, F(x) = r_i\, \bar{F}(r_i) + \int_{r_i}^{1} \bar{F}(x)\, dx \tag{13}$$

and a similar expression with r_i replaced by ρ. Substituting (13) into (11) we obtain:

$$v_i = r_i\, \bar{F}(r_i) + \int_{r_i}^{1} \bar{F}(x)\, dx + q\, v_{i+1}\, F(r_i) \tag{14}$$

It is our objective to determine the value of r_i which maximizes this expression.

Compare equation (14) with the expression of the value of the game if ρ were known:

$$v = \rho\, \bar{F}(\rho) + \int_{\rho}^{1} \bar{F}(x)\, dx + q\, v\, F(\rho). \tag{15}$$

Forming the difference of $v_i - V$ we obtain:

$$v_i - V = r\,\bar{F}(r) - \rho F(\rho) - \int_\rho^r \bar{F}(x)\,dx + q\,(v_{i+1}\,\bar{F}(r) - v\,F(\rho)) \quad (16)$$

where the integral $\int_\rho^r \bar{F}(x)\,dx$ is to be treated as $-\int_r^\rho \bar{F}(x)\,dx$ if r is less than ρ. Now this integral can be approximated as follows:

$$\int_\rho^r \bar{F}(x)\,dx \simeq \bar{F}(\rho)\,(r-\rho) - \tfrac{1}{2}\,f(\rho)\,(r-\rho)^2 . \quad (17)$$

Note also that $r\,F(r) - \rho\,F(\rho)$ may be replaced by:

$$r\,\bar{F}(r) - r\,\bar{F}(\rho) + r\,\bar{F}(\rho) - \rho\bar{F}(\rho) \quad (18)$$

and, upon rearranging and using $F(x) = 1 - F(x)$, by:

$$-r\,(F(r) - F(\rho)) + (r - \rho)) + (r - \rho)\,\bar{F}(\rho). \quad (19)$$

Similarly we can replace $q\,(v_{i+1}\,F(r) - v\,F(\rho))$ by:

$$q\,v_{i+1}\,(F(r) - F(\rho)) + q\,F(\rho)\,(v_{i+1} - v). \quad (20)$$

Substituting these expressions into equation (16) and cancelling terms we obtain:

$$v_i - v \simeq (q\,v_{j+1} - r)\,(F(r) - F(\rho)) + \tfrac{1}{2}\,f(\rho)\,(r - \rho)^2 + q\,F(\rho)\,(v_{i+1} - v) \quad (21)$$

Transferring the right most part to the left hand side, multiplying by g and replacing $F(r) - F(\rho)$ by the approximation:

$$f(\rho)\,(r - \rho) \tag{22}$$

we obtain:

$$q\,(v_i - v) - q^2\,F(\rho)\,(v_{i+1} - V) \simeq q\,(q\,v_{i+1} - r)\,f(\rho)\,(r - \rho) + \tfrac{1}{2}\,q\,f(\rho)\,(v - \rho)^2 \tag{23}$$

This can be simplified to:

$$q\,(v_i - v) - q^2\,F(\rho)\,(v_{i+1} - v) \simeq -\tfrac{1}{2}\,q\,f(\rho)(r - \rho)\,[\,(r - \rho) + 2\,(\rho - q\,v_{i+1})\,] \tag{24}$$

Now suppose we substitute the estimates $F_i(x)$ and $f_i(x)$ for the functions $F(x)$ and $F(x)$. We can then solve numerically the following equation, derived from (15) for an estimate of ρ which we will call ρ_i:

$$\underset{\rho_i}{\text{Max}}\ v = \left[\rho_i\,\overline{F}_i(\rho_i) + \int_{\rho_i}^{1} \overline{F}_i\,(x)\,dx\right] \div \left[1 - q\,F_i\,(\rho_i)\right] \tag{25}$$

where, of course, ρ_i is between zero and one. The estimate ρ_i will have a sampling distribution.

The value of r_i is set relative to ρ_i; let us define:

$$r_i = \rho_i + m_i \tag{26}$$

where m_i will be determined below. There will be a derivative sampling error in r_i. Let us observe that:

$$\Sigma(r_i) = q\, v_{i+1} \tag{27}$$

where $\Sigma(\cdot)$ denotes expected value and

$$p = q\, v. \tag{28}$$

These are necessary conditions for the threshold and estimated threshold to be in equilibrium. The conditions mean that one is indifferent to whether he waits or transits when the stopping rule is just satisified as an equality.

Let us define a new variable called d_i as follows

$$d_i = q(v_i - v) = \Sigma(r_{i-1}) - \rho \tag{29}$$

Now the value $\Sigma(r_{i-1})$ will approach ρ as i increases and d_i will consequently approach 0. In terms of d_i, equation (24) becomes

$$d_i - q\, F(\rho)\, d_{i+1} \simeq - \tfrac{1}{2} q\, f(\rho)\, [(r - \rho)^2 - 2\,(r - \rho)\, d_{i+1}] \tag{30}$$

If we now take expectation with regard to r on the right side of this equation we find

$$d_i - q\, F(\rho)\, d_{i+1} \simeq - \tfrac{1}{2} q\, f(\rho)\, [m_i^{\,2} + \sigma_i^{\,2} - 2\, m_i\, d_{i+1}] \tag{31}$$

where σ_i^2 is the sampling of r_i. Since we wish to maximize v_i and, consequently, d_i, we will maximize $2\, m_i\, d_{i+1} - m_i^2$ by setting $m_i = d_{i+1}$. If we assume that r is normally distributed, then the correct choice for the bias for r is thus

$$r \text{ distributed as } N(\rho + d_{i+1}, \sigma^2) \tag{32}$$

In passing, note that

$$v_i = v + \frac{1}{2} f(\rho)\,(d_{i+1}^2 - c_i^2) + qF(\rho)\, d_{i+1} \tag{33}$$

or

$$v_i = v + \frac{1}{2} f(\rho)\, [(q v_{i+1} - \rho)^2 - \sigma_i^2] + qf(\rho)\,(qv_{i+1} - \rho) \tag{34}$$

c. Application

It is assumed that, in practice, the following conditions are likely to obtain:

(1) Some information will be available on $F(\cdot)$. The form of the distribution could be known, and there may be a priori distributions on the parameters.

Also, some information on the x_i - in particular the statistics $\overline{x_i}$ and s_i - may have already been accumulated.

The "estimated" form of $F(\cdot)$ at the current stage of the game will be denoted by $F_b(\cdot)$ ("b" for Bayesian).

We shall be looking ahead at the next n stages of the game. $F_b(\cdot)$ will be used in place of $F(\cdot)$ for this purpose, in computing optimal strategy. This notation is distinct from the notation $F_n(\cdot)$ which referred to the current best estimate of $F(\cdot)$ in the stage-by-stage analysis of the game.

(2) We are content to "look ahead" a finite number of stages, n say, and to assume that at the n^{th} stage we have

$$v_{n+1} = v, \quad r_{n+1} = \rho, \quad m_n = d_{n+1} = qv_{n+1} - \rho = 0 . \tag{35}$$

(3) The variance of the estimate ρ_n at stage n, σ_n^2, can be estimated in advance. This estimate may be derived from previous experience; from sampling studies making use of $F_b(\cdot)$; or from other sources. We take

$$\sigma_{n+1} = 0 . \tag{36}$$

The procedure is then as follows. We shall determine strategy for the current stage (Stage 1) by looking ahead n stages. This strategy may then be updated, or recalculated, when the next value of x, namely x_2, is determined, provided that we have survived to Stage 2.

We first calculate estimates for ρ, v from equation (25) as:

$$\hat{v} = \sup_{\rho} \frac{\rho \overline{F}_b(\rho) + \int_{\rho}^{1} \overline{F}_b(r)\, d_n}{1 - qF_b(\rho)} \tag{37}$$

where the supremum is attained for

$$\rho = \hat{\rho}. \tag{38}$$

We now apply equation (34) to stage n, and work backwards until we reach Stage 1. $F_b(\cdot)$ is used for $F(\cdot)$, etc., and $\hat{\rho}$ is used for ρ.

$$(v_{n+1} = \hat{v}) \tag{39}$$

$$v_n = \hat{v} + (1/2)\, f_b\, (\hat{\rho})\, [(q\, v_n - \hat{\rho})^2 - \sigma^2_{n-1}]$$

$$+ q\, f\, (\hat{\rho})\, (q\, v_n - \hat{\rho}) \tag{40}$$

$$\vdots$$

$$v_i = \hat{v} + (1/2)\, f_b (\hat{\rho})\, [(q\, v_{i+1} - \hat{\rho})^2 - \sigma^2_i]$$

$$+ q\, f(\hat{\rho})\, (q\, v_{i+1} - \hat{\rho}) \tag{41}$$

$$\vdots$$

$$v_1 = \hat{v} + (1/2)\, f_b (\hat{\rho})\, [q\, v_2 - \hat{\rho})^2 - \sigma^1_2]$$

$$+ q\, f(\hat{\rho})\, q\, v_2 - \hat{\rho})\,. \tag{42}$$

Once v_2 is known, the correct choice for r_1, is

$$r_1 = \hat{\rho} + d_2 = \hat{\rho} + (q\, v_2 - \hat{\rho}) = q\, v_2\,. \tag{43}$$

d. References

Chow, Y. S. and Robbins, H. E. (1963) "On the Values associated stochastic sequence," Proc. Fifth Berkeley Symp. Math. Stat. Prob.

Gilbert, John P., and Mosteller, F., March (1966) "Recognizing the Maximum of a Sequence," J. A. S. A.

D. INSTANTANEOUS PROBABILITY DETECTION

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D. INSTANTANEOUS PROBABILITY DETECTION

1. Introduction

As indicated in the introductory chapter, a satisfactory model of sonar detection is an essential aspect of analyzing submarine encounters with patrolling vessels. In the course of our studies we have examined several approaches. One prevalent method is referred to as the "λ, Δ, σ, model" from the occurrence of three parameters represented by these three Greek letters in the equations of the model. Because of its popularity, we review in the next sections that model and the procedures used for fitting it to observed data. Another method that is less widely used is the instantaneous detection model. It is conjectured, however, that this model better explains the times to detection observed in actual exercises. In section 3 we will describe the instantaneous detection model in detail and in section 4 we will illustrate the procedures for fitting the model to observations from exercises. Finally, in section 5, we will derive maximum likelihood estimators and confidence regions for the parameters.

2. The λ, Δ, σ Model

We will first describe a method for determining detection probability that has been extensively used. Let us begin by assuming the geometry and other parameters (orientation, depth, speed, etc.) of the submarine transit are fixed. For example, these can be taken from an actual exercise.

In Figure D-1, the dotted line represents the expected signal excess of the returned ping from the submarine over the background noise, when the submarine is within range. However, it is hypothesized for this model that the actual signal excess is randomly perturbed from the expected signal excess. Such perturbation is presumed to account for the observed variations in detection performance. The solid line in the figure depicts a possible actual signal excess distribution.

This method models the variation from the expected signal excess in a particular manner. Let epochs of time $t_1, t_2, \ldots$ be selected along the t axis by drawing from an exponential distribution with parameter λ. At each time t_i the difference in expected and actual is presumed to change. The amount of the new difference is found by drawing a value from the Normal distribution $N(0, \sigma^2)$. This value is superimposed (added to) the theoretical dotted line to obtain the solid line. Detection is then said to occur whenever the solid line crosses the threshold Δ and the result of the model is a binary decision as to whether detection occurs. To obtain the probability of detection, the whole process is repeated many times drawing different time epochs, and signal jumps, each time. The resulting number of detections, divided by sample size, yields the probability of detection.

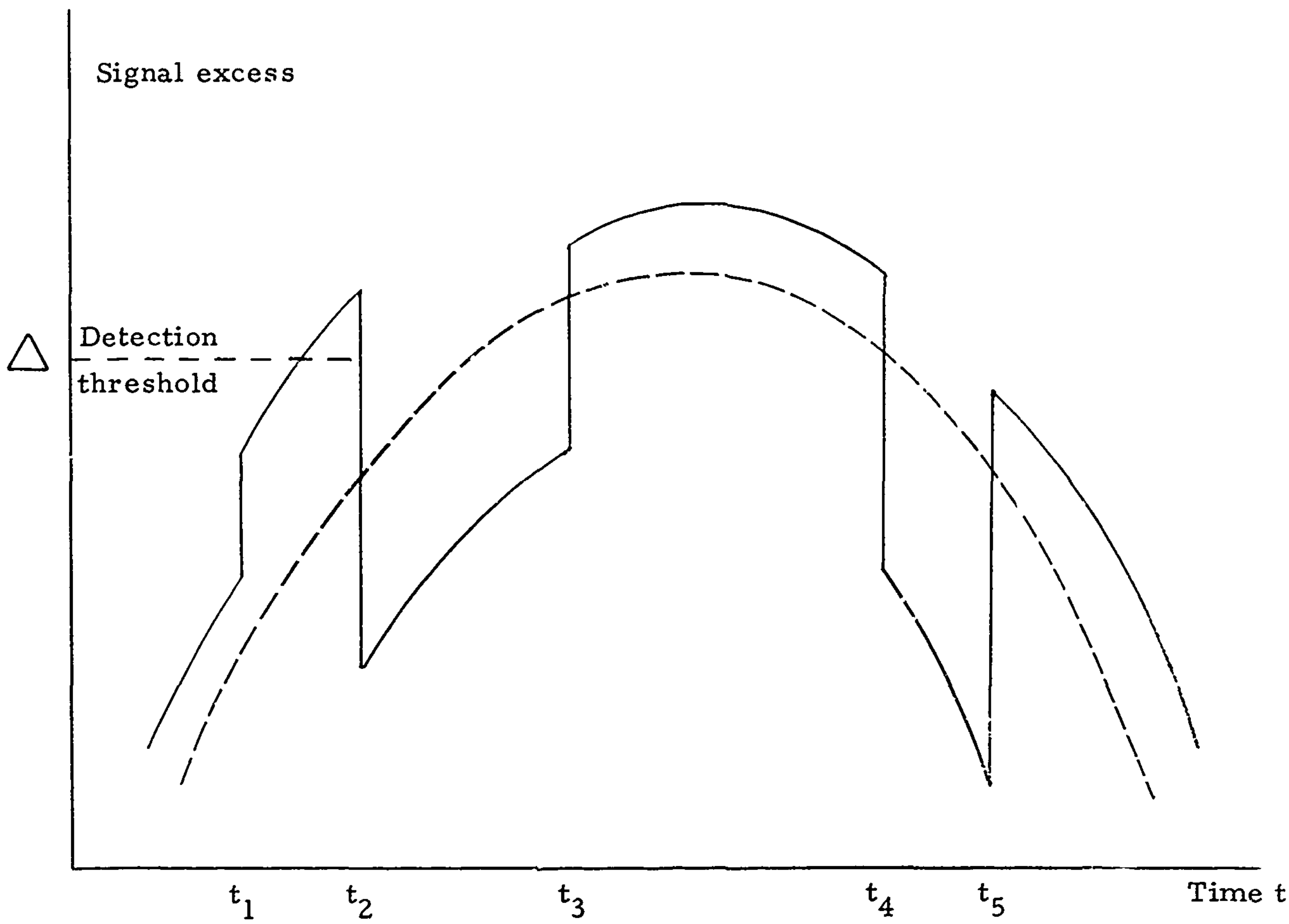

Figure D-1. The λ, Δ, σ method for detection probability.

We can now vary the parameters λ, Δ, and σ and obtain different values for the probability of detection.

This whole process is repeated for a group of attempted penetrations. Various tests - some quite complicated - have been proposed for determining the set of parameter $(\lambda, \Delta, \sigma)$ values which provide the best fit.

3. The Instantaneous Detectio.. ..ate Model

This model differs in one important aspect: a signal excess over the threshold Δ does not imply that detection occurs. Rather the significant factor is the time that each positive signal excess persists. This is accomplished in the following manner.

As shown in Figure D-2, we again start with a "theoretical" predicted signal excess curve, shown as a solid line in the figure. The signal excess, over a stated threshold, is indicated by the vertical distance E(t). Since a positive value for E(t) no longer implies instant detection, we calculate an instantaneous probability of detection function, as follows.

Let us assume that at any constant level of E(t) the probability of detection is the same in any interval of time from (t) to $(t + \Delta t)$ for small Δt. This is equivalent to assuming that the detection rate is Poisson distributed with a parameter λ called the instantaneous detection rate. The probability that no detections occur is

$$e^{-\lambda}.$$

In the more general case that λ is a function of time, the probability of no detection in the period when the excess is positive is:

$$\exp\left[-\int_{\lambda > 0} \lambda(t)\, dt\right].$$

The signal excess E(t) and the instantaneous detection rate $\lambda(t)$ are closely related. We have hypothesized in this work that a quadratic relationship is sufficiently accurage. A linear relationship was tried in Reference 1 and did not appear adequate.

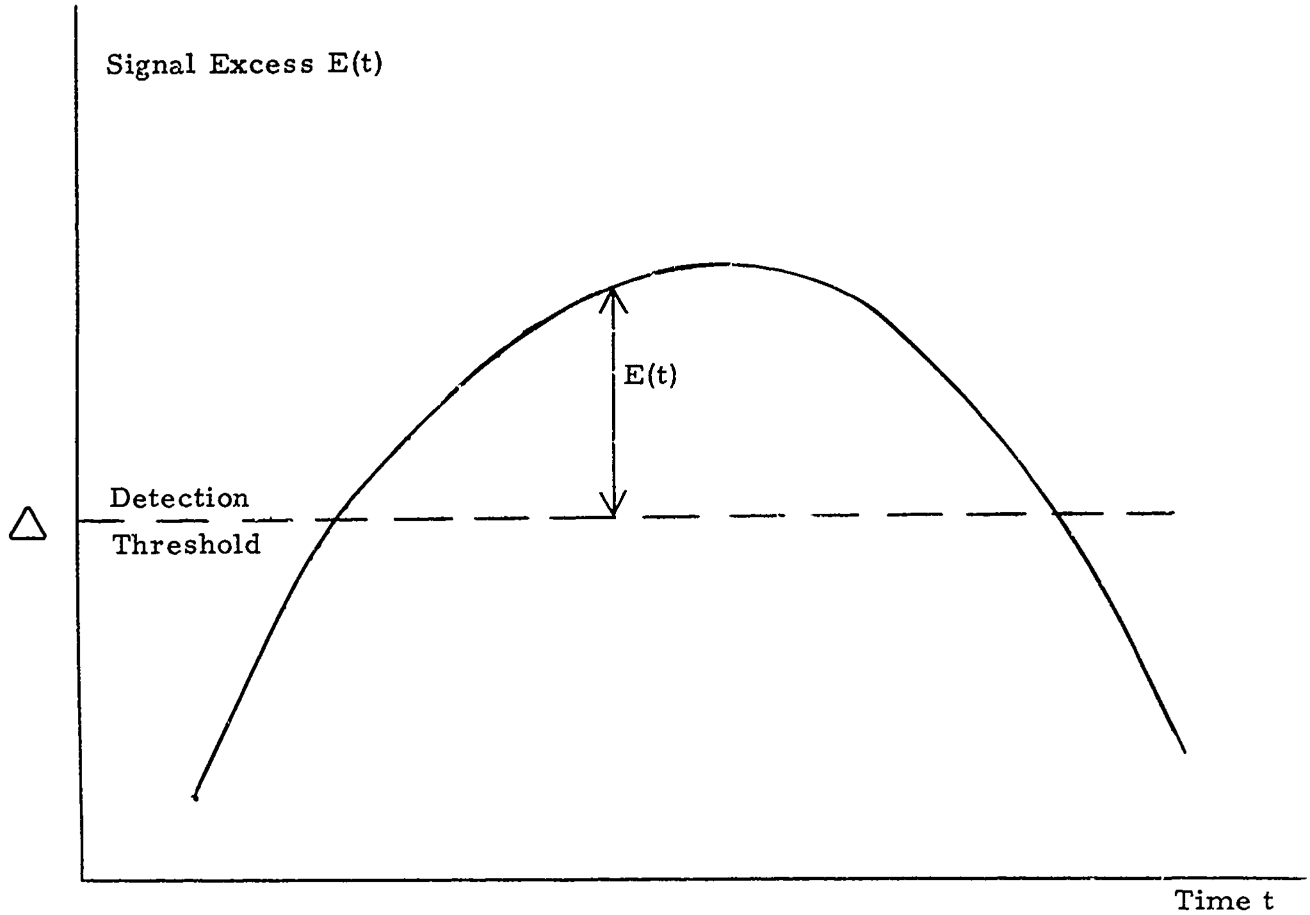

Figure D-2. Signal Excess for the Instantaneous Detection Model

Thus, let us define the relationship:

$$\lambda(t) = A \cdot E(t) + B \cdot E^2(t)$$

where A and B are parameters. In figures 3 and 4 we illustrate respectively the relation of $\lambda(t)$ to $E(t)$ and a sample $\lambda(t)$ curve which might result from applying the $\lambda(t)$ curve in figure 3 to the $E(t)$ curve in figure 2.

The area under the curve in Figure 4 is equal to $\int \lambda(t)dt$. This area is measured in decibel-minutes, abbreviated db.-min., and the model is frequently referred to as the db.-min. model (read "dee-bee-minutes model").

In the following sections, a method for estimating these parameters is given, and also (specifically for the linear case when B = 0) a method for obtaining confidence intervals for these parameters is given.

In predicting the probability of detection for future ASW exercises, these confidence intervals are used as follows. They provide, in a sense, an idea of the "fuzziness" associated with detection model parameters. The size of the intervals can be used to directly calculate, not only the expected number of detections in a planned group of exercises, but also the range of the number of detections (or, a lower limit for the number), that would be reasonable taking into account the difficulty in estimating environmental conditions, temperature gradients of the water, etc.

These items are calculated directly rather than estimated by means of repeated Monte Carlo runs. This produces a more

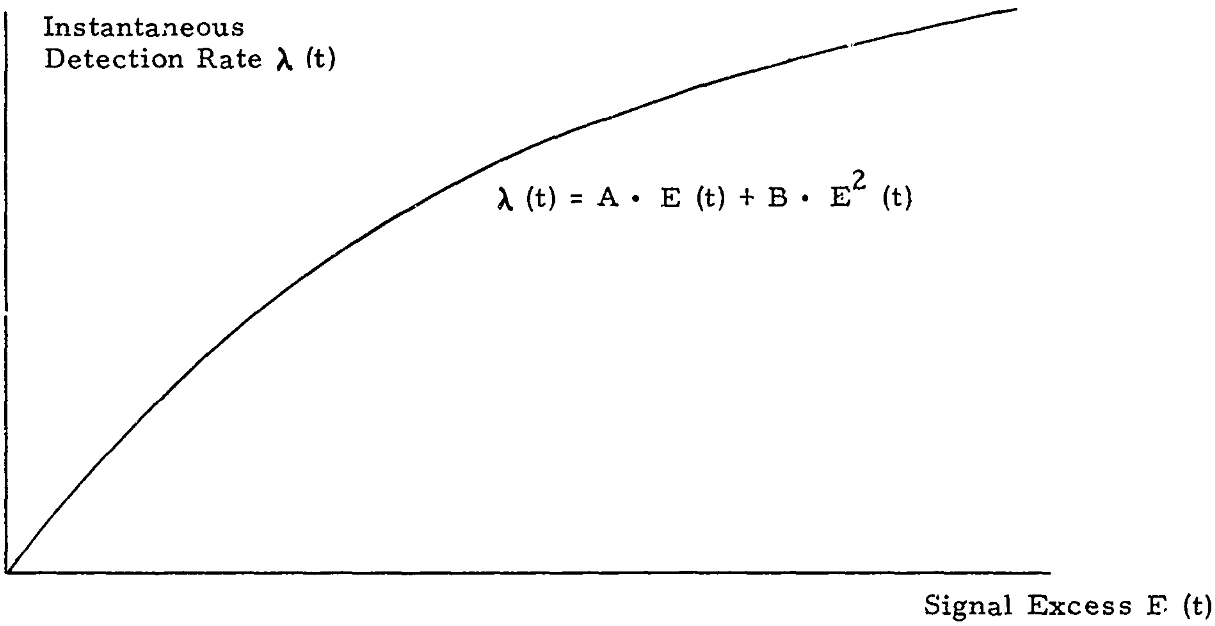

Figure D-3. Relation of Instantaneous Detection Rate to Signal Excess

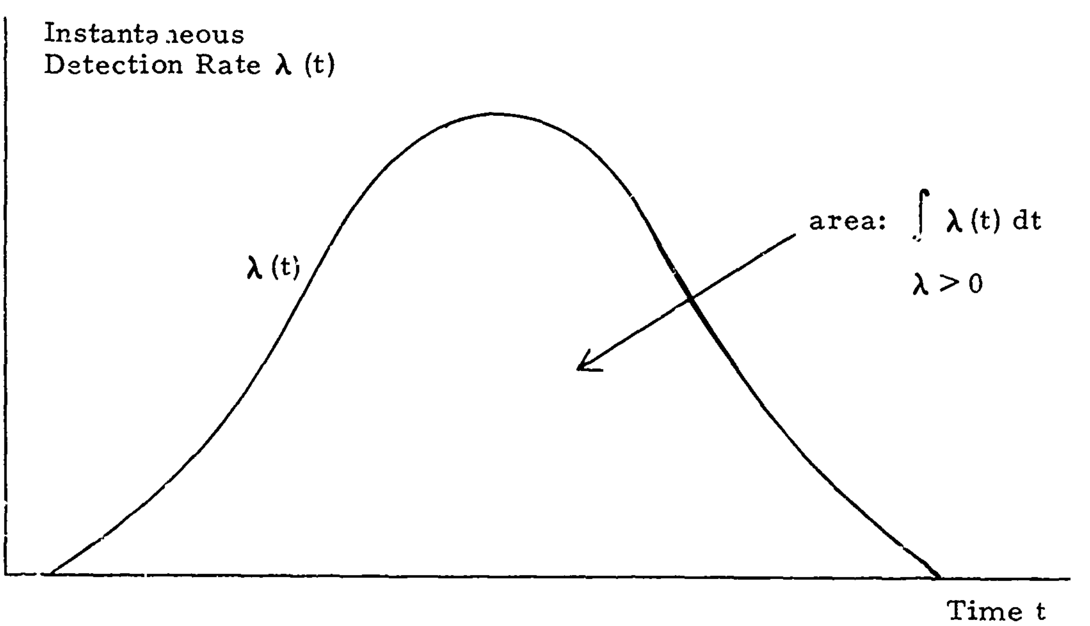

Figure D-4. Instantaneous Detection Rate for the Instantaneous Detection Rate Model reliable and easily used model.

In addition, in the last section, there will be found methods for testing goodness of fit of the models proposed here.

4. Fitting the Instantaneous Detection Rate Model to Observed Data

A certain amount of data has been collected on the distribution of time to first detection in destroyer vs. submarine games. It is conjectured that an instantaneous detection rate model may adequately explain the variation in times to first detection. A "brute-force" maximum likelihood method is described for obtaining estimates of model parameters, and joint confidence regions. These estimates would be used in estimating probabilities of detection in barrier (or convoy screen) models, and in stating such probabilities with a required degree of confidence.

In the following discussion we shall let S represent the transiting submarine and D the destroyer or other sonar platform. Let us assume that we have the following data.

(i) The positions of S and D at the start (beginning of exercise)

(ii) The courses of S and D throughout the exercise

(iii) Time of first (valid) detection of S by D, if any. If contact lost, times of subsequent re-detections.

We assume that, conditional on no detection by time t, the probability of detection in the interval $(t, t + dt)$ is $\lambda(t)\,dt$ (the usual instantaneous rate assumption). In fact, following the usual noise propagation assumptions, suppose initially that

$$\lambda \sim \max[0, \{L_S(v) - L_D(u) + N_{DI} - N_{RD}\} - k \log r]$$

where

$L_D(u)$ = "self-noise" of Destroyer speed u

$L_S(v)$ = radiated noise of Submarine at speed v

N_{DI} = directivity factor

N_{RD} = "recognition differential" necessary to distinguish [with probability 50%] the Submarine from background noise. [If the period of observation is infinitely long.]

r = distance between S and D

k = exponent of "noise law" (usually taken as 2)

(Conversion factors for db and $\log_e$ have been omitted.)

We may conveniently write

$$\lambda \sim \text{function } (v, u, DI, RD) - k \log r \tag{2}$$

In the following v will denote the velocity of S relative to D.

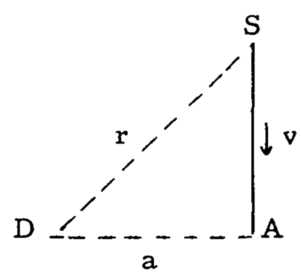

Figure D-5. Geometry of Approach Path

For approximate straight-line motion let A be the closest point of approach. Then:

$$r^2 = a^2 + v^2 t^2 \tag{3}$$

$$2 \log r = \log (a^2 + v^2 t^2).$$

λ can thus be expressed directly in terms of time. And, in fact, this expression (for a straight line segment of the relative course) can be integrated exactly over time. This procedure is described in detail in another chapter in this (Chapter B).

Thus, provided in approximate motion on straight line segments, we obtain an exact expression for

$$P(\text{no detection}) = \exp\left[-\int_{t:\lambda(t)>0} \lambda(t)\,dt\right] \qquad (4)$$

$$P(\text{detection occurs at } t^*) = \lambda(t^*) \exp\left[-\int_{t:\lambda(t)\geq 0}^{t^*} \lambda(t)\,dt\right]dt^* \qquad (5)$$

5. Maximum Likelihood Estimators and Confidence Regions for (A, B)

Equation (4), or (5), with $t = t^*$, represents the likelihood that first detection occurs at t^* in a particular trial. The likelihood can be summed over t^*'s, taken from different trials. For trials where $t = \infty$ (no detection occurred) the expression

$$\exp\left[-\int_{t_1}^{\infty} \lambda(t)\, dt \right]. \tag{6}$$

should be used.

We obtain an overall likelihood function

$$L(A, B; t^*) = P(t^* \mid A, B). \tag{7}$$

The values of A and B which maximize this expression can be found by a steepest ascent procedure. Furthermore, using approximate methods, we can determine a "confidence region" for A, B, namely a region

$$C_{95\%} = (A_1 \leq A \leq A_2,\ B_1 \leq B \leq B_2) \tag{8}$$

such that the mass of the likelihood function contained in $C_{95\%}$ is 0.95. These confidence bounds on A, B can then be used in computing the overall probability of detection in a proposed scenario, and of providing a lower confidence limit on the number of detections out of 100 trials (say).

The foregoing procedure closely follows the methodology in the reference [3], Singpurwalla. The field of application in the latter is reliability theory, but the same instantaneous detection rate function applies, and the application is very similar.

Alternatively, the following "standard" procedure may be sufficiently good, provided the total number of observations is sufficiently large. It is known (Kendall, p. 42) that maximum likelihood estimators are asymptotically efficient. In our case the maximum likelihood estimators for A and B are jointly normally distributed, asymptotically, and the variances and covariance can be obtained by taking the 2nd derivative of the log likelihood function. This can certainly be done analytically in our case, although the computation will be somewhat tedious.

Thus we can use the bivariate Normal distribution and use standard tables to pick out our 95% confidence region for (A, B).

Even better, we can store the parameters of this bivariate distribution. Then, in any concrete situation, such as determining the probability of detection when a submarine attempts to penetrate a patrol barrier using some stated patrol strategy, we can proceed as follows.

We compute the probability of detection using an (approximate) analytical formula, and assuming A and B known exactly. Then, introduce the joint (asymptotic) distribution for A and B, and integrate out A and B. This will be the maximum likelihood estimate of probability of detection. Next, assuming that A and B are present in the probability of detection formula in some simple analytical form such as A/B or $(A^2 + B^2)^{1/2}$, we can obtain the distribution of this form and obtain confidence limits directly on the probability of detection.

The question arises whether the quadratic "failure-rate" model for $\lambda(t)$ is adequate to explain observed variations in first detection time.

The cumulative distribution function of t, time to first detection, is given by

$$1 - \exp\left[- \int_{t_1}^{t} \lambda(t)\, dt \right]. \tag{9}$$

The values of this function should, if our model is correct, be random observations from the rectangular distribution on [0, 1]. The Kolmogoroff test can be used to verify this.

However, we may consider our model "sufficiently good" if any anomalies from this rectangular distribution are small, and no significant improvement can be made on the fit by dividing the data in sub-classes (e.g. trials with a sea-state of 5 or lower; with an S-speed of 5-7 knots; etc.).

In similar situations, Kolmogorov tests have been applied, but heavy use has been made of Monte-Carlo simulation to provide tests of significance. These tests, and significance levels, must be calculated anew for each new situation.

In contrast, recent work by Lilliefors [2] indicates that more general significance levels can be obtained in conjunction with the Chi-squared test. His paper also suggests the optimal number of intervals to use (at least six).

This paper does not deal specifically with truncated observations of the type we encounter, but the following obvious modification suggests itself.

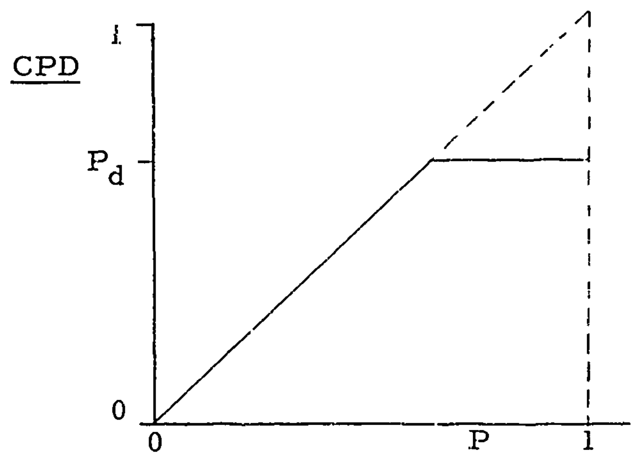

Figure D-6. Value of CPD at which detection occurs (if at all)

First, all the predicted cumulative probability of detection functions are normalized, as shown in the figure. In general, Pr (detection) will increase up to an asymptotic value, denoted by P_d.

The observed value of the cumulative probability distribution (CPD) at which detection occurs is denoted by p. If no detection occurs, we arbitrarily draw the value p from a uniform distribution on the interval $[P_d, 1]$.

Suppose that the recommended number of intervals is six. The following figure illustrates the observed p's, in histogram form, from a series of runs (n in number).

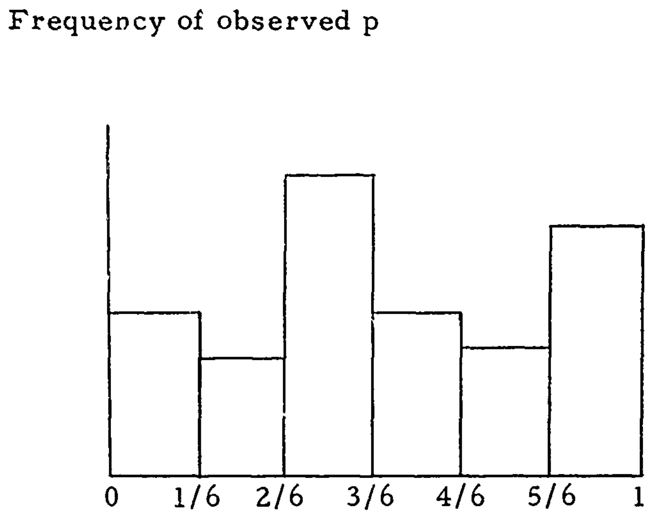

Figure D-7. Hypothetical Frequency of Observed p.

The Chi-square test is then applied in the usual way, making use of the significance levels supplied in Lilliefors' paper. Note, in that paper, that appropriate attention is paid to the number of parameters of the "db-minutes" model which are estimated <u>from the data</u>.

As far as is known (from the available literature) no other goodness-of-fit test currently in use, or proposed, for detection models has this feature. In view of the small sample sizes experienced, the importance of this feature is obvious.

6. References

[1] COMDESDEVGRU2, ltrser 076, August 3rd, 1970, "Analysis of Data on Destroyer ASW Screening Mission."

[2] Lilliefors, Hubert W., "Some Monte-Carlo results on the Chi-square goodness-of-fit statistic when parameters are estimated from the sample." Private Communication.

[3] Singpurwalla, Nozer D., "A problem in accelerated life-testing," June 26, 1970; Institute for Management Science, The George Washington Univ. [Submitted, J. Am. Stat. Assoc.]

[4] Kendall, M., "Advanced Theory of Statistics."

E. THE DISTRIBUTION OF THE TRANSIT POINT IN A SUBMARINE VS. DESTROYER GAME

Preface

The distribution of the Transit Point in a Submarine vs. Destroyer Game, Split-the-Gap Strategy

1.0 Introduction

1.1 Comments on possible applications, and related work

2.0 Assumptions

3.0 Mathematical statement of the problem

4.0 Results

4.1 Three destroyers

4.2 Two destroyers

5.0 Analysis

5.1 Three destroyers

5.2 Two destroyers

References

E. THE DISTRIBUTION OF THE TRANSIT POINT IN A SUBMARINE VS. DESTROYER GAME

Preface

This chapter contains the text of an article submitted during the project to the Naval Logistics Research Quarterly. Since no decision has been reached on publishing it, it is included here in its entirety.

The subject of the paper, tne distribution of a submarine's transit point, arose tangentially to the general study of submarine-destroyer encounters. He represents an abstract theoretical analysis of the probability distribution of the location of the midpoint of the larger gap. In so far as this theoretical distribution approaches the actual distribution occurring in practice, the results have several useful applications. One which appears to require some attention on the part of the Navy is to use sonobuoys or other silent sonar systems distributed in proportion to the gap distribution. The objective would be to minimize the maximum probability of penetration anywhere along a screen or barrier.

The article follows.

THE DISTRIBUTION OF THE TRANSIT POINT IN A SUBMARINE VS. DESTROYER GAME, SPLIT-THE-GAP STRATEGY.[1]

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MATHEMATICA, Bethesda, Md.

Abstract

Destroyers patrol randomly and independently along three segments of a straight-line barrier. A submarine takes one, and only one, look at their positions, and picks a transit point by splitting the larger gap. What is the probability distribution of this point ?

1.0 Introduction

Three destroyers patrol a line (stationary, in the barrier case; advancing with constant speed in the convoy escort case).

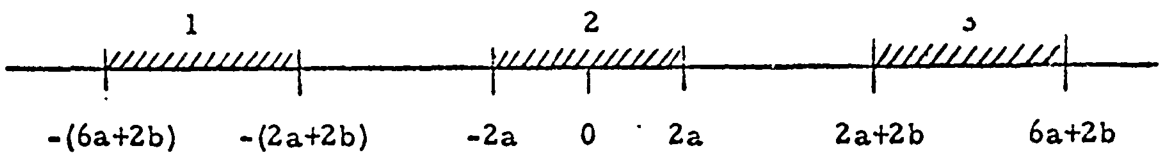

Fig. 1: Patrol Segments, General Case

The patrol segments are specified in Fig. 1 and are supposed of equal length, with equal gaps between adjacent segments.

[1]Work performed under Contract N00014-70-C-0307, Office of Naval Research. The author acknowledges helpful discussions with Mr J, Randolph Simpson, ONR, and Mr Peter Perkins, TRW Systems.

A submarine views the three destroyers instantaneously, and once only, and determines their positions (but not their directions of motion). The submarine chooses the larger of the two gaps and takes the mid-point M of the gap as his transit point. The purpose of this appendix is to determine the distribution of this point M. The simpler, two destroyer case is solved also.

1.1 Comments on possible applications, and related work

(a) The theoretical distributions could be compared with empirical distributions of transit points (after suitable normalization, to take account of different barrier dimensions). Unfortunately, the latter distributions appear in classified publications.

(b) Suppose that the submarine crosses the barrier line a fixed time T after it takes a look at the destroyers' position. This 'lag' T could be assumed sufficiently large so that the destroyer positions at T would be independently distributed of their positions at time zero. Alternatively, the new positions could be assumed correlated with the old positions, or some specific patrol pattern could be simulated during the lag T.

The probability of detection (destroyer on submarine), PD, could then be calculated using cookie-cutter, or instantaneous probability ([2], p. 506) methods.

(c) In a game theory situation, the destroyers could attempt to improve PD by modifying the distribution of their position within the respective segments. Techniques which are applicable to this problem can be found in [1].

On the other side, the submarine could attempt to lower PD by basing his choice of transit point on the specific positions of all three destroyers, instead of just splitting the larger gap.

(d) We could elaborate the single-look assumption, by allowing the submarine to acquire some information on destroyer positions when (if ever) he gets close enough to one or more of them, and allow some modification of submarine course thereafter.

2.0 Assumptions

We assume the destroyers patrol randomly and independently in their segments, and in such a way that(e.g.) destroyer 1's position at an arbitrary time is uniformly distributed in the segment $(-(6a+2b), -(2a+2b))$.

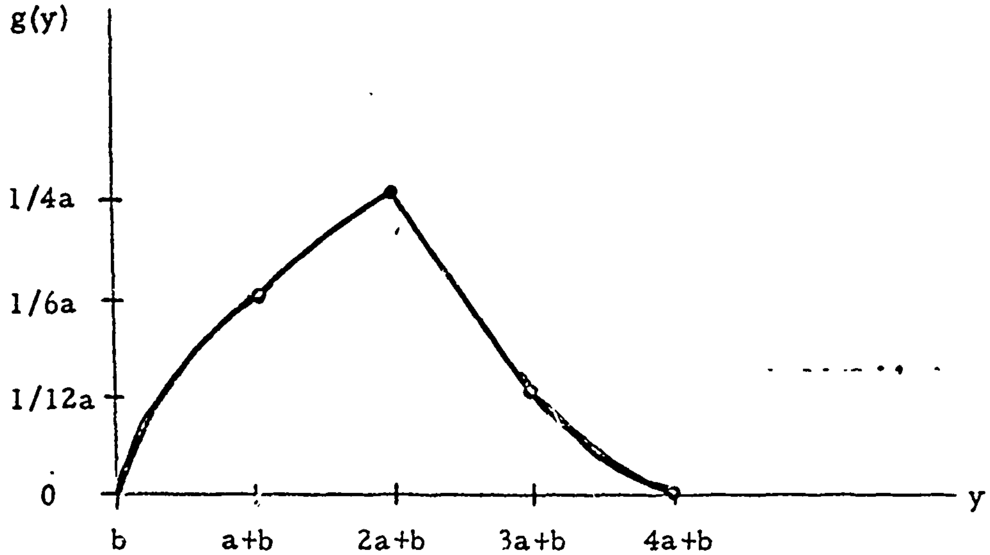

Fig. 2: g(y), p.d.f. for gap-split point, y (3 destroyer case)

4.2 Two Destroyers

If only destroyers 2 and 3 are involved, we obtain.

	Range for y	p.d.f., g(y)
(4.2)	b, 2a+b	$(y-b) / 4a^2$
	2a+b, 4a+b	$[4a-(y-b)] / 4a^2$

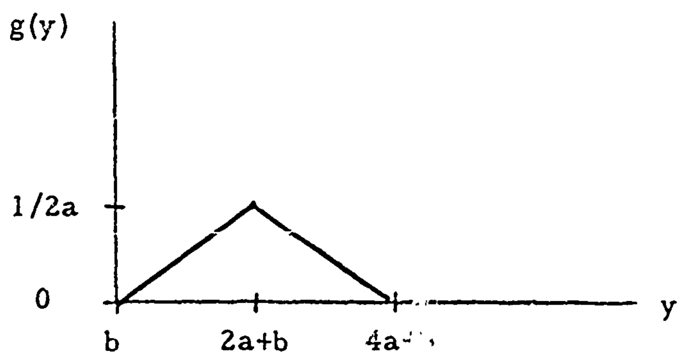

Fig. 3: g(y), p.d.f. for gap-split point, y (2 destroyer case)

3.0 Mathematical Statement of Problem

x_i, $i=1, 3$ are independent random variables uniformly drawn from the ranges shown in Fig. 1..

y is defined as the mean of x_2, x_3 if $x_3 - x_2 \leq x_2 - x_1$ otherwise as the mean of x_1, x_2.

4.0 Results

4.1 Three Destroyers

The p.d.f. for y, $g(y)$, is continuous:

$$
(4.1)\quad
\begin{cases}
b,\ a+b & (y-b)\,[3a-(y-b)] \,/\, 12a^3 \\
a+b,\ 2a+b & [(y-b)+a] \,/\, 12a^2 \\
2a+b,\ 3a+b & [7a-2(y-b)] \,/\, 12a^2 \\
3a+b,\ 4a+b & [(y-b)-4a]^2 \,/\, 12a^3
\end{cases}
$$

(Columns: Range for y; p.d.f., g(y))

Also, $g(b) = g(4a+b) = 0$

$g(a+b) = 1/6a$

$g(2a+b) = 1/4a$

$g(3a+b) = 1/12a$

5.0 Analysis

5.1 Three Destroyers

It is sufficient to consider the problem with "the gaps closed up." A simple transformation, $y \to (y-b)$, with appropriate changes for the range, then yields the result in Sec. 4.

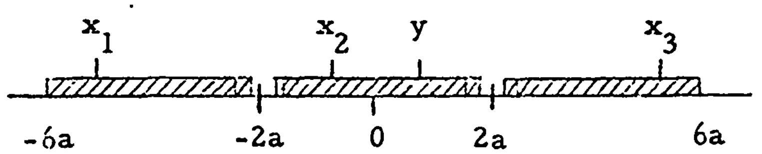

Fig. 4: Simplified Situation

In the following work, assume that y falls in the right-hand side of the figure (i.e. $y > 0$). The resulting distribution will then be conditional on this event, and, using symmetry, we can easily obtain the true p.d.f. for y.

To save excessive repetition of the factor (1/4a), take the p.d.f.'s for the x_i as 1, for the moment. The notation $(,)$ is to be taken as referring to the closed interval.

Fix y in $(0,4a)$.

The "allowable" range for x_2 [i.e., that for which there exists an x_3 yielding the stated y] is obtained as follows.

$$x_3 = 2y - x_2, \in (2a, 6a)$$

i.e. $x_2 \in (2y - 6a, 2y - 2a)$. Also, of course, $x_2 \in (-2a, 2a)$.

Now, for y to be the gap-split point, we must have

$$x_1 > x_2 - (2y - x_2) = 3x_2 - 2y .$$

Also, $x_1 \in (-6a, -2a)$.

The following figure indicates the "allowable" range for x_1 as a function of x_2. The position of the points A, D can change relative to B, C, depending on the value of y originally fixed.

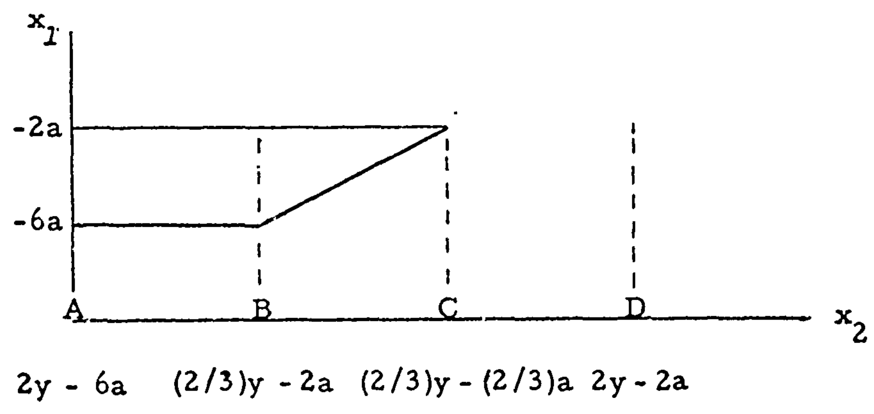

Fig. 5: Allowable Range for x_1

[We have made use of the following:

$$3x_2 - 2y = \begin{cases} -6a \\ -2a \end{cases} \quad \text{implies} \quad x_2 = \begin{cases} (2/3)y - 2a \\ (2/3)y - (2/3)a \end{cases} . \;]$$

Still keeping y fixed, we now integrate over the permitted ranges (dependent on y). Two types of integral arise, as shown:

(i) $\quad y \in (2a,3a)$: $\displaystyle\int_{2(y-3a)}^{(2/3)(y-3a)} 4a \cdot dx_2 = (16/3)a(3a-y)$

$\quad y \in (0,2a)$: $\displaystyle\int_{-2a}^{(2/3)(y-3a)} 4a \cdot dx_2 = (8/3)ay$

(ii) $\quad y \in (a,3a)$: $\displaystyle\int_{(2/3)(y-3a)}^{(2/3)(y-a)} (2y-3x_2-2a)\, dx_2 = (8/3)a^2$

$\quad y \in (0,a)$: $\displaystyle\int_{(2/3)(y-3a)}^{2(y-a)} (2y-3x_2-2a)\, dx_2 = (8/3)y(2a-y)$

$\quad y \in (3a,4a)$: $\displaystyle\int_{2(y-3a)}^{(2/3)(y-a)} (2y-3x_2-2a)\, dx_2 = (8/3)(y-4a)^2$

Performing the integration over y, we obtain

(5.1) $\quad a^3\,[8/3 + 16/3 + 16/3 + 16/9 + 8/9] = 16a^3$.

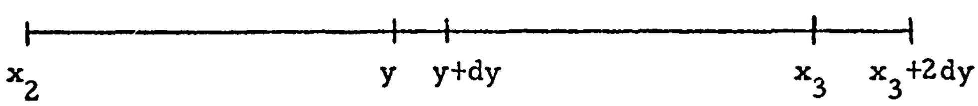

Fig. 6: Allowable Increment for x_3

Consideration of Fig. 6 shows the correct p.d.f. for x_5 is $2/4a = (1/2a)$; the p.d.f. for x_2, x_3: $(1/4a)$. We thus apply a factor $(1/2a) \cdot (1/4a)^2 = (1/32a^3)$ to (5.1), obtaining 1/2. The other (1/2) of the p.d.f. for y occurs for $y \in (-4a, 0)$.

Combining the expressions in (i),(ii), and introducing the factor $(1/32a^3)$, we obtain the result in Sec. 4.

5.2 Two Destroyers

If destroyers 2 and 3 only are involved, the corresponding integrals are

$$2a < y \leq 4a: \int_{2y-6a}^{2a} 1 \cdot dx_2 = 8a - 2y$$

$$0 \leq y \leq 2a: \int_{-2a}^{2y-2a} 1 \cdot dx_2 = 2y$$

The "correct" factor is $(1/8a^2)$, by an argument similar to that in 5.1, and we obtain the result in Sec. 4.2.

REFERENCES

[1] "A study of Optimal Patrol and Transit Strategies in a Rectangular Barrier Zone, 'using mathematical games", Final Report, November 4, 1965, ONR Contract No. N0nr-4937(00), MATHEMATICA, Princeton, N. J.

[2] "Theory of Search II", Bernard O. Koopman, OPERATIONS RESEARCH, October, 1956, pp. 503-531.